Steffen Fröhlich

Surfaces in Euclidean Spaces

Contents

Part I
Theory of Surfaces

Chapter 1
Immersions in $\mathbb{R}^{n+2}$

This first chapter introduces the reader into basic important concepts of differential geometry of twodimensional immersions in $\mathbb{R}^{n+2}$, particularly including tangential and normal spaces along with the concept of orthonormal normal frames and orthogonal transformations between them. Furthermore we introduce the concept of conformal parameters which provide a surface representation most suitable for our purposes. Next we define the three fundamental forms, and a discussion about geometrical nature tensors and its transformation behaviour concludes this first chapter.

1.1 Regular surfaces

The objects of all our present investigations are *twodimensional immersions of disc-type in parametric form* with trace in Euclidean spaces $\mathbb{R}^{n+2}$ with a natural number $n \geq 1$. More precisely, we consider *two-dimensional vector-valued mappings*

$$X = X(u,v) = \big(x^1(u,v),\ldots,x^{n+2}(u,v)\big) \in C^4(B,\mathbb{R}^{n+2}),$$

defined on the *closed unit disc*

$$B := \big\{(u,v) \in \mathbb{R}^2 \,:\, u^2+v^2 \leq 1\big\} \in \mathbb{R}^2 .$$

Additionally we set

$$\mathring{B} := \big\{(u,v) \in \mathbb{R}^2 \,:\, u^2+v^2 < 1\big\}$$

for the *open unit disc*, i.e. $\mathring{B}$ denotes the *interior* of B, and we write

$$\partial B := \big\{(u,v) \in \mathbb{R}^2 \,:\, u^2+v^2 = 1\big\}$$

for its boundary. Many of our following considerations require *simply connected and smoothly bounded domains of definition.* On the other hand, restricting to surfaces defined only on the closed unit disc B can justified on account of Riemann's mapping theorem.[1]

[1] If $\Omega \subset \mathbb{R}$ is a simply connected and open domain which is not all of $\mathbb{C}$ then there exists a bijective and holomorphic mapping from Ω onto the open unit disc.

Furthermore we assume that all these mappings X are *regular in the differential geometric sense,* i.e. we require they represent *immersions* in the following sense

$$\operatorname{rank} DX \equiv \operatorname{rank} \begin{pmatrix} x_u^1 & x_v^1 \\ \vdots & \vdots \\ x_u^{n+2} & x_v^{n+2} \end{pmatrix} = 2 \quad \text{in } B \tag{1.1}$$

for the rank of the *Jacobian matrix* $DX \in \mathbb{R}^{n+2,2}$, where the lower indices u and v denote the partial derivatives w.r.t. to the respective parameters. This condition ensures that at each point $w \in B$ there exist two linearly independent *tangential vectors* X_u and X_v of the mapping X (simultaneously its partial derivatives w.r.t. u and v) spanning the *twodimensional tangential space* at this point $w \in B$, i.e.

$$\mathfrak{T}_X(w) := \operatorname{Span}\{X_u(w), X_v(w)\} \cong \mathbb{R}^2, \quad w \in B. \tag{1.2}$$

In particular, this leads us to the decomposition

$$\mathbb{R}^n = \mathfrak{T}_X(w) \oplus \mathfrak{N}_X(w)$$

with the n-dimensional *normal space*

$$\mathfrak{N}_X(w) := \{Z \in \mathbb{R}^{n+2} : Z \cdot X_u(w) = Z \cdot X_v(w) = 0\} \tag{1.3}$$

at each point $w \in B$, where

$$X \cdot Y = \sum_{i=1}^{n+2} x^i y^i$$

denotes the inner product between two vectors X and Y.

Finally, the regularity condition (1.1) ensures that at each point $w \in B$ the *area element* W of the surface is positive, i.e. it holds

$$W := \sqrt{(X_u \cdot X_u)(X_v \cdot X_v) - (X_u \cdot X_v)^2} > 0 \quad \text{in } B. \tag{1.4}$$

Example 1.1. The twodimensional mapping

$$X(u,v) = (u, v, u^2 - v^2, 2uv) \colon B \longrightarrow \mathbb{R}^4$$

represents an immersion since the tangential vectors

$$X_u = (1, 0, 2u, 2v), \quad X_v = (0, 1, -2v, 2u)$$

are always non-parallel and non-vanishing. In particular, it represents a *surface graph* defined over the $[x,y]$-plane with coordinates $x = u$ and $y = v$. As we will see later, X is also a *minimal graph,* and it will serve as a typical example in various situations of our analysis.

1.2 Remarks on branch points

We want to exclude *irregular points* $w \in B$ where the regularity condition (1.1) is violated. An example of such a *non-immersed* surface is given by the *complexified Neil's parabola*

$$X(w) = (w^2, w^3), \quad w = u + iv \in \mathbb{C},$$

representing a *minimal surface* with a so-called *branch point* at $w = 0$ where[2]

$$X_u(0,0) = 0 = X_v(0,0).$$

Using a calibration argument we will show in chapter 12 that X *has smallest area compared with all surfaces spanning the same boundary as* X. In contrast to this example, a *minimizing surface in three-dimensional Euclidean space* $\mathbb{R}^3$ *is indeed immersed* (see e.g. Osserman's monograph [128]), at least in the interior, which shows that the geometry of surfaces in $\mathbb{R}^{n+2}$ with arbitrary dimension turns out to be more intricate than the classical differential geometry.

One word about Neil's parabola

$$c(t) = (t^2, t^3)$$

found by William Neil (1637-1670) in 1657: It represents that curve along which a particle, descending under gravity, describes equal vertical spacings within equal times (see e.g. Eric Weinstein's resource *http://mathworld.wolfram.com* [163]).

Finally, the *Henneberg surface,* parametrically given by

$$\begin{aligned}
x^1(u,v) &= 2\sinh u \cos v - \frac{2}{3}\sinh(3u)\cos(3v), \\
x^2(u,v) &= 2\sinh u \sin v - \frac{2}{3}\sinh(3u)\sin(3v), \\
x^3(u,v) &= 2\cosh(2u)\cos(2v),
\end{aligned}$$

is an example of a non-immersed minimal surface in $\mathbb{R}^3$ with a branch point at the origin $(u,v) = (0,0)$.

1.3 Orthonormal normal frames

Our regularity assumptions ensure also that for any $w \in B$ we can choose n linearly independent, orthogonal unit normal vectors N_σ, $\sigma = 1,\ldots,n$, spanning the normal space $\mathfrak{N}_X(w)$ at $w \in B$.

---

[2] Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we will eventually write $X(w)$ instead of $X(u,v)$ etc.

Due to the contractibility of the domain of definition B we can extend this set to a C^3-regular *orthonormal normal frame*

$$N(w) = \big(N_1(w),\ldots,N_n(w)\big) \in \mathbb{R}^{n\times(n+2)}$$

moving smoothly along the whole surface and satisfying

$$\begin{aligned} &N_\sigma(w)\cdot X_u(w) = 0 = N_\sigma(w)\cdot X_v(w)\,, \\ &N_\sigma(w)\cdot N_\vartheta(w) = \delta_{\sigma\vartheta} := \begin{cases} 1, & \text{if } \sigma=\vartheta \\ 0, & \text{if } \sigma\neq\vartheta \end{cases} \quad \text{for all } w\in B \end{aligned} \tag{1.5}$$

with the Kronecker symbol $\delta_{\sigma\vartheta}$.

Definition 1.1. A matrix $N\in\mathbb{R}^{n\times(n+2)}$ with these properties is called an *orthonormal normal frame* of X, shortly: *ONF.*

Example 1.2. The unit normal vector N for an immersion $X\colon B\to\mathbb{R}^3$ is defined as

$$N(w) := \frac{X_u(w)\times X_v(w)}{|X_u(w)\times X_v(w)|}\,,\quad w\in B,$$

with the usual vector product $\times$ in threedimensional Euclidean space. In this case we simply identify the ONF with the vector $N(w)$.

Example 1.3. The *Euler unit normal vectors*

$$\begin{aligned} \widehat{N}_1 &:= \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,(-\varphi_x,-\varphi_y,1,0), \\ \widehat{N}_2 &:= \frac{1}{\sqrt{1+|\nabla\psi|^2}}\,(-\psi_x,-\psi_y,0,1) \end{aligned} \tag{1.6}$$

are orthogonal to the *surface graph*

$$X(x,y) = \big(x,y,\varphi(x,y),\psi(x,y)\big)\in\mathbb{R}^4\,,\quad (x,y)\in\Omega\subset\mathbb{R}^2\,,$$

with two smooth functions φ and ψ. Namely, for $\sigma=1,2$ we immediately compute

$$\begin{aligned} \widehat{N}_\sigma\cdot X_x &= \widehat{N}_\sigma\cdot(1,0,\varphi_x,\psi_x) \,=\, 0, \\ \widehat{N}_\sigma\cdot X_y &= \widehat{N}_\sigma\cdot(0,1,\varphi_y,\psi_y) \,=\, 0. \end{aligned}$$

But note that *in general* $\widehat{N}_1$ *and* $\widehat{N}_2$ *are not orthogonal to each other:*

$$\widehat{N}_1\cdot\widehat{N}_2 = \frac{\nabla\varphi\cdot\nabla\psi}{\sqrt{(1+|\nabla\varphi|^2)(1+|\nabla\psi|^2)}} \neq 0.$$

Thus we define new unit vectors

$$\widetilde{N}_1 := \widehat{N}_1\,, \quad \widetilde{N}_2 := \frac{\widehat{N}_2 - (\widehat{N}_1 \cdot \widehat{N}_2)\widehat{N}_1}{|\widehat{N}_2 - (\widehat{N}_1 \cdot \widehat{N}_2)\widehat{N}_1|}$$

taking into account that

$$|\widehat{N}_2 - (\widehat{N}_1 \cdot \widehat{N}_2)\widehat{N}_1|^2 = 1 - \frac{(\nabla\varphi \cdot \nabla\psi)^2}{(1+|\nabla\varphi|^2)(1+|\nabla\psi|^2)} > 0.$$

This new system $(\widetilde{N}_1, \widetilde{N}_2)$ now forms an ONF for the graph X.

Example 1.4. Let

$$\Phi(u,v) := \varphi(u,v) + i\psi(u,v) \in \mathbb{C}$$

be a holomorphic function satisfying Cauchy-Riemann equations

$$\varphi_x = \psi_y\,, \quad \varphi_y = -\psi_x \quad \text{in } \Omega \subset \mathbb{R}^2\,.$$

Then the surface graph $\big(x, y, \varphi(x,y), \psi(x,y)\big)$ fulfills

$$X_x^2 \,=\, 1 + \varphi_x^2 + \psi_x^2 \,=\, 1 + \psi_y^2 + \varphi_y^2 \,=\, X_y^2\,,$$
$$X_x \cdot X_y \,=\, \varphi_x\varphi_y + \psi_x\psi_y \,=\, 0.$$

We say X *is a conformally parametrized surface;* see section 1.6 below for an exact definition of conformality. Furthermore, we compute

$$\nabla\varphi \cdot \nabla\psi = \varphi_x\psi_x + \varphi_y\psi_y = -\varphi_x\varphi_y + \varphi_y\varphi_x = 0,$$

where $\nabla\varphi = (\varphi_x, \varphi_y)$ denotes the Euclidean gradient of φ. We conclude that the Euler normals $\widehat{N}_1$ and $\widehat{N}_2$ from (1.6) form an orthonormal normal frame.

1.4 Rotation of normal frames

Quantities representing inner geometric properties of a surface do not depend on the choice of an ONF. Rather we can say an ONF is just a mathematical tool to describe the behaviour of a surface in terms of dynamical quantities. Hence we may refer to it as a *moving frame.*

Thus in a concrete situation we are supposed to prove *independence of the geometric quantities from the special choice of an ONF.* In particular, this concerns all curvature quantities describing the inner geometry of a surface as well as its embedding into the space.

For this purpose we now present a simple algorithm to transform any given ONF N into a new ONF $\widetilde{N}$ *conserving the orientation.*

Namely, let us consider matrix-valued orthogonal mappings

$$\mathbf{R} = (r_{\sigma\omega})_{\sigma,\omega=1,\dots,n} \in C^3(B, SO_n)$$

satisfying the properties

$$\sum_{\sigma=1}^{n} r_{\sigma\omega}(w)^2 = 1, \quad \sum_{\sigma=1}^{n} r_{\omega\sigma}(w)^2 = 1 \quad \text{for } \omega = 1,\dots,n,$$
$$\sum_{\sigma=1}^{n} r_{\sigma\omega}(w) r_{\sigma\omega'}(w) = 0, \quad \sum_{\sigma=1}^{n} r_{\omega\sigma}(w) r_{\omega'\sigma}(w) = 0 \quad \text{for } \omega \neq \omega'$$

as well as

$$\det \mathbf{R}(w) = 1 \quad \text{for all } w \in B.$$

This latter property characterizes the conservation of orientation. For simplicity, we will only focus on *rotations as transformation mappings*. Now the promised transformation between the two ONF is given by

$$\widetilde{N}_\sigma(w) = \sum_{\omega=1}^{n} r_{\sigma\omega}(w) N_\omega(w), \quad \sigma = 1,\dots,n. \tag{1.7}$$

Note that there hold

$$|\widetilde{N}_\sigma|^2 = \sum_{\omega=1}^{n} r_{\sigma\omega}^2 = 1$$

as well as

$$\widetilde{N}_\sigma \cdot \widetilde{N}_\vartheta = \sum_{\omega,\omega'=1}^{n} r_{\sigma\omega} r_{\vartheta\omega'} \delta_{\omega\omega'} = \sum_{\omega=1}^{n} r_{\sigma\omega} r_{\vartheta\omega} = \delta_{\sigma\vartheta}$$

in B for all $\sigma, \vartheta = 1,\dots,n$.

Example 1.5. In case $n = 2$, i.e. if X is immersed into $\mathbb{R}^4$, we choose the orthogonal mapping

$$\mathbf{R} := \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{1.8}$$

with a rotation angle $\varphi \in \mathbb{R}$. In particular, we have

$$\widetilde{N}_1 = \cos\varphi\, N_1 + \sin\varphi\, N_2\,,$$
$$\widetilde{N}_2 = -\sin\varphi\, N_1 + \cos\varphi\, N_2\,.$$

Example 1.6. In case $n = 3$ we may introduce the *Euler rotations* generated by the three matrices

$$\mathbf{R}_1 := \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}$$

as well as

$$\mathbf{R}_3 := \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformation works as follows: First we rotate by φ about the z-axis, then by ψ about the x-axis, and, finally, by ϑ about the z-axis again. The effective rotation is then described by (see e.g. Funk [68])

$$\mathbf{R} = \mathbf{R}_3 \circ \mathbf{R}_2 \circ \mathbf{R}_1 . \tag{1.9}$$

1.5 The fundamental forms

Differential geometry is essentially built up on the three fundamental forms. Already Gauss [70] was aware of its special importance.

Let us start with the

Definition 1.2. The *first fundamental form* $I(X) \in \mathbb{R}^{2\times 2}$ of X is defined by

$$I(X) = (g_{ij})_{i,j=1,2}, \quad g_{ij} := X_{u^i} \cdot X_{u^j} \tag{1.10}$$

setting $u^1 := u$ and $u^2 := v$.

Note that the definition of this quadratic form does not depend on the codimension. *Rather it is simply induced by the Euclidean metric of the embedding space* $\mathbb{R}^{n+2}$. To clarify, let us denote by

$$ds^2 = dx_1^2 + \ldots + dx_{n+2}^2 \tag{1.11}$$

the standard line element of the space $\mathbb{R}^{n+2}$ with $x_1, \ldots, x_{n+2}$ being its Euclidean coordinates. Embedding the surface $X = X(u,v)$ into the Euclidean space $\mathbb{R}^{n+2}$ means, from that formal point of view,

$$\begin{aligned} ds^2 &= (x_{1,u}\,du + x_{1,v}\,dv)^2 + \ldots + (x_{n+2,u}\,du + x_{n+2,v}\,dv)^2 \\ &= (X_u \cdot X_u)\,du^2 + 2(X_u \cdot X_v)\,du dv + (X_v \cdot X_v)\,dv^2 \\ &= g_{11}\,du^2 + 2g_{12}\,du dv + g_{22}\,dv^2 = \sum_{i,j=1}^{2} g_{ij}\,du^i du^j . \end{aligned}$$

Obviously it holds

$$W = \sqrt{g_{11}g_{22} - g_{12}^2} = \sqrt{\det I(X)}$$

for the area element W.

Definition 1.3. The *second* and the *third fundamental form* w.r.t. any unit normal vector $N\colon B \to \mathbb{R}^{n+2}$ are

$$\begin{aligned} I\!I_N(X) &= (L_{N,ij})_{i,j=1,2}\,, \quad L_{N,ij} := -X_{u^i} \cdot N_{u^j} = X_{u^i u^j} \cdot N\,, \\ I\!I\!I_N(X) &= (e_{N,ij})_{i,j=1,2}\,, \quad e_{N,ij} := N_{u^i} \cdot N_{u^j}\,. \end{aligned} \tag{1.12}$$

Note that these forms now depend on the chosen unit normal vector N.

The first and the second fundamental form for surfaces in $\mathbb{R}^3$ were already introduced by Gauss [70]. In this special case of one codimension, the coefficients of the second fundamental form are also denoted by L, M and N, a terminology which goes back to Reinhold Hoppe (compare Strubecker [152]).

The geometric meaning of the first fundamental form is contained in the following result from elementary differential geometry which we do not prove here.

Theorem 1.1. *The knowledge of the coefficients g_{ij}, $i,j = 1,2,3$, enables us to answer all questions about the metrical properties of a surface, in particular problems concerning length of and angles between curves on it, or area of pieces of it.*

Example 1.7. Consider a curve $c\colon I \subset \mathbb{R} \to B$ and its spatial image $X \circ c(t) \in \mathbb{R}^3$ on the surface X. Let $c(t) = (u^1(t), u^2(t))$. Then its length is given by the integral

$$\mathscr{L}(c_i) = \int_I \sqrt{\sum_{k,\ell=1}^{2} g_{k\ell} \frac{du_i^k}{dt} \frac{du_i^\ell}{dt}}\, dt\,.$$

Furthermore, the angle α between two curves $X \circ c_1$ and $X \circ c_2$ with $c_1 = (u_1, v_1)$ and $c_2 = (u_2, v_2)$, and which intersect at $c_1(t) = c_2(t) = w \in B$, is

$$\begin{aligned} \cos\alpha &= \frac{c_1' \cdot c_2'}{|c_1'||c_2'|} \\ &= \frac{g_{11}\dot u_1 \dot u_2 + g_{12}\{\dot u_1 \dot v_2 + \dot u_2 \dot v_1\} + g_{22}\dot v_1 \dot v_2}{\sqrt{g_{11}\dot u_1^2 + 2g_{12}\dot u_1 \dot v_1 + g_{22}\dot v_1^2}\sqrt{g_{11}\dot u_2^2 + 2g_{12}\dot u_2 \dot v_2 + g_{22}\dot v_2^2}}\,. \end{aligned}$$

1.6 Parameter systems

In this section we want to present so-called conformal parameter systems and geodesic polar coordinates. Other possible parametrization are discussed in later chapters.

Conformal parameters

Definition 1.4. The immersion $X:\ B \to \mathbb{R}^{n+2}$ is said to be *conformally parametrized* if the *conformality relations*

$$X_u \cdot X_u = W = X_v \cdot X_v\,, \quad X_u \cdot X_v = 0 \quad \text{in } B \tag{1.13}$$

with the area element W from (1.4) as conformal factor are satisfied.

The special meaning of these coordinates is that they *diagonalize the Riemannian line element* such that we get

$$ds^2 := X_u^2\,du^2 + 2X_u \cdot X_v^t\,dudv + X_v^2\,du^2 = W(du^2 + dv^2)$$

on the whole disc B. Here we say ds^2 is of *Riemannian type* if it holds

$$X_u^2 X_v^2 - (X_u \cdot X_v)^2 > 0 \quad \text{in } B.$$

This is always true in view of our regularity assumptions.

Introducting conformal parameters into a Riemannian metric is justified by the following results.

Proposition 1.1. *(Sauvigny [142], 1999)*
Assume that the coefficients a, b and c of the Riemannian metric

$$ds^2 = a\,du^2 + 2b\,dudv + c\,dv^2$$

are of class $C^{1+\alpha}(B,\mathbb{R})$ with $\alpha \in (0,1)$. Then there is a conformal parameter system $(u,v) \in B$.

The regularity condition required here is satisfied in our situation because we suppose $g_{ij} \in C^3(B,\mathbb{R})$. While Sauvigny's result holds *in the large,* i.e. on the whole closed disc B, further stronger results hold *in the small.* One example is the following.

Proposition 1.2. *(Chern [28], 1955)*
Assume that the coefficients of the Riemannian metric

$$ds^2 = a\,du^2 + 2b\,dudv + c\,dv^2$$

are Hölder continuous in B. Then for every point $w \in \mathring{B}$ there exists an open neighborhood over which the surface can be pararmetrized conformally.

Example 1.8. As we already know, the mapping

$$X(w) = (w, w^2) \in \mathbb{R}^4$$

fulfills the conformality relations.

Example 1.9. The *catenoid* is a rotationally minimal surface which can be represented using conformal parameters as

$$X(u,v) = (\cosh u \cos v, \cosh u \sin v, u) \in \mathbb{R}^3 .$$

The *conformally parametrized minimal helicoid*

$$\widetilde{X}(u,v) = (-\sinh u \sin v, \sinh u \cos v, -v) \in \mathbb{R}^3$$

is closely related to this catenoid: Namley, let $\lambda \in [0, \frac{\pi}{2}]$ be a real parameter, then catenoid and helicoid can be transformed isometrically into each other by means of the mapping

$$X_\lambda(u,v) := X(u,v)\cos\lambda + \widetilde{X}(u,v)\sin\lambda .$$

In particular, X_0 represents the catenoid, while $X_{\frac{\pi}{2}}$ equals the helicoid. This isometry is a special case of a general theorem due to Bour on the isometrically deforming a rotationally symmetric surface into a ruled surface (see e.g. Strubecker [152]).

Remark 1.1. Note that if only $X \in C^1(B, \mathbb{R}^{n+2})$ *it is in general not true* that X can be parametrized conformally, see e.g. Chern, Hartman and Wintner [29] for a detailed discussion along with various counterexamples.

Geodesic polar coordinates

For a detailed introduction of the so-called *exponential map, geodesic discs* and *geodesic polar coordinates* we refer the reader to elementary textbooks on differential geometry, e.g. Blaschke and Leichtweiss [15], Klingenberg [106], or Laugwitz [111]. Here we just want to provide some important identities needed laters.

Assume that the immersion $X\colon B \to \mathbb{R}^{n+2}$ (or a part of it) is given as a *geodesic disc* $\mathfrak{B}_r(X_0)$ of geodesic radius $r > 0$ and with center $X_0 \in \mathbb{R}^{n+2}$. Using *geodesic polar coordinates* $(\rho, \varphi) \in [0,r] \times [0,2\pi]$ we can rewrite the mapping $X = X(u,v)$ into the new form

$$Z = Z(\rho,\varphi)\colon [0,r] \times [0,2\pi] \longrightarrow \mathbb{R}^{n+2} .$$

Following e.g. Blaschke and Leichtweiss [15], §79, the new line element ds_P^2 reads

$$ds_P^2 = Z_\rho^2\, d\rho^2 + 2Z_\rho \cdot Z_\varphi\, d\rho\, d\varphi + Z_\varphi^2\, d\varphi^2 = d\rho^2 + P(\rho,\varphi)\, d\varphi^2 .$$

with a function $P \in C^1((0,r] \times [0,2\pi), \mathbb{R})$ satisfying

$$P(\rho,\varphi) > 0 \quad \text{for all } (\rho,\varphi) \in (0,r] \times [0,2\pi)$$

as well as

$$\lim_{\rho \to 0_+} P(\rho,\varphi) = 0, \quad \lim_{\rho \to 0_+} \frac{\partial}{\partial \rho} \sqrt{P(\rho,\varphi)} = 1 \quad \text{for all } \varphi \in [0,2\pi).$$

1.7 Parameter transformations. Tensors

We want to restrict our investigations to the following special class of parameter transformations.

Definition 1.5. The C^4-regular parameter transformation

$$v^\alpha \mapsto u^i(v^\alpha), \quad i=1,2,\ \alpha=1,2,$$

is called *of regularity class* $\mathfrak{P}$, shortly: $u^i(v^\alpha)\in\mathfrak{P}$, if it is positively oriented, i.e.

$$\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)}>0$$

for its Jacobian, and if it does not effect the orientation of the unit normal vectors of the ONF N.

We compute the transformed derivatives of the surface vector

$$X_{u^i}=\sum_{\lambda=1}^{2}X_{v^\lambda}\,v^\lambda_{u^i}=\sum_{\lambda=1}^{2}\Lambda_i^\lambda X_{v^\lambda} \quad \text{setting } \Lambda_i^\lambda:=v^\lambda_{u^i}$$

for $i=1,2$. The tangential vectors X_{u^i} of the surface X are *inner geometrical objects.* From our point of view, they are parametrized vector-valued mappings. Thus it is important to compare the transformation behaviour of any of our quantities with that of the X_{u^i} what finally leads us to the concept of *tensors.*

Let us start with the transformation behaviour of the g_{ij} :

$$g_{ij}=X_{u^i}\cdot X_{u^j}=\sum_{\kappa,\lambda=1}^{2}\Lambda_i^\kappa\Lambda_j^\lambda X_{v^\kappa}\cdot X_{v^\lambda}=\sum_{\kappa,\lambda=1}^{2}\Lambda_i^\kappa\Lambda_j^\lambda g_{\kappa\lambda}$$

Next let $\overline{\Lambda}_\kappa^i$ denote the coefficients of the inverse transformation satisfying

$$\sum_{\kappa=1}^{2}\Lambda_i^\kappa\overline{\Lambda}_\kappa^j=\delta_i^j\,, \quad \sum_{i=1}^{2}\Lambda_i^\kappa\overline{\Lambda}_\lambda^i=\delta_\lambda^\kappa$$

with the Kronecker symbols δ_i^j and δ_λ^κ. It follows the inverse transformation rule

$$\begin{aligned}\sum_{i,j=1}^{2}g_{ij}\overline{\Lambda}_\mu^i\overline{\Lambda}_\nu^j&=\sum_{i,j=1}^{2}\sum_{\kappa,\lambda=1}^{2}(\Lambda_i^\kappa\overline{\Lambda}_\mu^i)(\Lambda_j^\lambda\overline{\Lambda}_\nu^j)g_{\kappa\lambda}\\&=\sum_{\kappa,\lambda=1}^{2}\delta_\mu^\kappa\delta_\nu^\lambda g_{\kappa\lambda}=g_{\mu\kappa}\,.\end{aligned}$$

As an immediate consequence we obtain the

Corollary 1.1. *Let $v^\alpha \mapsto u^i(v^\alpha)$ be a parameter transformation from class $\mathfrak{P}$, and denote by $W(u^1,u^2)$ and $W(v^1,v^2)$ the area elements of the surface $X\colon B \to \mathbb{R}^{n+2}$ w.r.t. u^i resp. v^α. Then it holds*

$$W(u^1,u^2) = (\Lambda_1^1\Lambda_2^2 - \Lambda_1^2\Lambda_2^1)W(v^1,v^2).$$

Proof. This follows from evaluating the expression

$$\begin{aligned} W(u^1,u^2)^2 = & \left(\sum_{\kappa,\lambda=1}^2 \Lambda_1^\kappa\Lambda_1^\lambda g_{\kappa\lambda}\right)\left(\sum_{\mu=1}^2 \Lambda_2^\mu\Lambda_2^\nu g_{\mu\nu}\right) \\ & - \left(\sum_{\kappa,\lambda=1}^2 \Lambda_1^\kappa\Lambda_2^\lambda g_{\kappa\lambda}\right)\left(\sum_{\mu,\nu=1}^2 \Lambda_1^\mu\Lambda_2^\nu g_{\mu\nu}\right) \end{aligned}$$

which proves the statement. □

There will also appear tensors of higher rank, for example the curvature tensors of surfaces.

Definition 1.6. Let $r \in \mathbb{N}$, $r \geq 1$. The function $a_{i_1\ldots i_r}$ is called a *covariant tensor of rank r*, or shortly a *r-covariant tensor* if and only if it transforms in the way

$$a_{i_1\ldots i_r} = \sum_{\mu_1,\ldots,\mu_r=1}^2 \Lambda_{i_1}^{\mu_1}\cdot\ldots\cdot\Lambda_{i_r}^{\mu_r} a_{\mu_1\ldots\mu_r}\,.$$

Beside these covariant components we need *contravariant tensors.* These forms are defined in terms of the inverse first fundamental form, i.e. in terms of the coefficients g^{ij} defined by

$$\sum_{j=1}^2 g_{ij}g^{jk} = \delta_i^k$$

with the Kronecker symbol δ_i^k.

Definition 1.7. Let $r \in \mathbb{N}$, $r \geq 1$. The *contravariant components* of the covariant tensor $a_{i_1\ldots i_s}$ are defined as

$$a^{i_1\ldots i_r} := \sum_{j_1,\ldots,j_r=1}^2 g^{i_1j_1}\cdot\ldots\cdot g^{i_rj_r} a_{j_1\ldots j_r}\,.$$

To compute the transformation behaviour of the coefficients $g^{k\ell}$ we first notice

$$g^{k\ell} = \sum_{i,j=1}^2 g_{ij}g^{ik}g^{j\ell}\,.$$

Then we calculate

$$\begin{aligned} g^{k\ell} &= \sum_{i,j=1}^{2} g_{ij} g^{ik} g^{j\ell} = \sum_{i,j=1}^{2} \sum_{\kappa,\lambda=1}^{2} (\Lambda_i^{\kappa} \Lambda_j^{\lambda} g_{\kappa\lambda}) g^{ik} g^{j\ell} \\ &= \sum_{i,j=1}^{2} \sum_{\kappa,\lambda=1}^{2} (\Lambda_i^{\kappa} g^{ik})(\Lambda_j^{\lambda} g^{j\ell}) g_{\kappa\lambda} \\ &= \sum_{i,j=1}^{2} \sum_{\kappa,\lambda=1}^{2} \sum_{\mu,\nu=1}^{2} (\Lambda_i^{\kappa} g^{ik})(\Lambda_j^{\lambda} g^{j\ell}) g_{\kappa\mu} g_{\lambda\nu} g^{\mu\nu} \\ &= \left(\sum_{i=1}^{2} \sum_{\kappa=1}^{2} \Lambda_i^{\kappa} g^{ik} g_{\kappa\mu} \right) \left(\sum_{j=1}^{2} \sum_{\lambda=1}^{2} \Lambda_j^{\lambda} g^{j\ell} g_{\lambda\nu} \right) g^{\mu\nu} = \sum_{\mu,\nu=1}^{2} \overline{\Lambda}_{\mu}^{k} \overline{\Lambda}_{\nu}^{\ell} g^{\mu\nu} \end{aligned}$$

where we take

$$\begin{aligned} \sum_{i=1}^{2} \sum_{\kappa=1}^{2} \Lambda_i^{\kappa} g^{ik} g_{\kappa\mu} &= \sum_{i=1}^{2} \sum_{\kappa=1}^{2} \Lambda_i^{\kappa} g^{ik} \left(\sum_{m,n=1}^{2} g_{mn} \overline{\Lambda}_{\kappa}^{m} \overline{\Lambda}_{\mu}^{n} \right) \\ &= \sum_{i=1}^{2} \sum_{m,n=1}^{2} \sum_{\kappa=1}^{2} \Lambda_i^{\kappa} \overline{\Lambda}_{\kappa}^{m} \overline{\Lambda}_{\mu}^{n} g^{ik} g_{mn} \\ &= \sum_{i=1}^{2} \sum_{m,n=1}^{2} \delta_i^m \overline{\Lambda}_{\mu}^{n} g^{ik} g_{mn} = \sum_{m,n=1}^{2} \overline{\Lambda}_{\mu}^{n} g^{mk} g_{mn} = \sum_{n=1}^{2} \overline{\Lambda}_{\mu}^{n} \delta_n^k = \overline{\Lambda}_{\mu}^{k} \end{aligned}$$

etc. into account. This gives rise to our next

Definition 1.8. Let $s \in \mathbb{N}$, $s \geq 1$. The function $a^{i_1 \dots i_s}$ is called a *contravariant tensor of rank s*, or shortly a *s-contravariant tensor* if and only if it transforms as

$$a^{i_1 \dots i_r} = \sum_{\mu_1,\dots,\mu_r=1}^{2} \overline{\Lambda}_{\mu_1}^{i_1} \cdot \ldots \cdot \overline{\Lambda}_{\mu_s}^{i_r} a^{\mu_1 \dots \mu_s} . \tag{1.14}$$

Finally, we need covariant *and* contraviant tensor components.

Definition 1.9. Let $r,s \in \mathbb{N}$, $r,s \geq 1$. The function $a_{i_1 \dots i_r}^{j_1 \dots j_s}$ is called a (r,s)-*tensor* if and only if it transforms as

$$a_{i_1 \dots i_r}^{j_1 \dots j_s} = \sum_{\mu_1,\dots,\mu_r=1}^{2} \sum_{\nu_1,\dots,\nu_s=1}^{2} \Lambda_{i_1}^{\mu_1} \cdot \ldots \cdot \Lambda_{i_r}^{\mu_r} \cdot \overline{\Lambda}_{\nu_1}^{j_1} \cdot \ldots \overline{\Lambda}_{\nu_s}^{j_s} a_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_s} . \tag{1.15}$$

For detailed treatises we refer the reader to textbooks on differential geometry, e.g. Bär [4], do Carmo [21], [22], Kühnel [109], or Raschewski [135].

1.8 Vector fields

Geometric quantities can also be expressed using *coordinate-independent vector fields.* Although we will mainly use special coordinate systems matching our special problems, we eventually formulate results also using geometric vector fields to emphasize their geometric importance.

In particular, covariant tensors are sort of *multi-valued linear mappings on the tangential space.* Let us consider the following example for illustration: The coefficients g_{ij} of the first fundamental form can be considered as the components of a bi-linear mapping

$$g\colon \mathfrak{T}_X(w)\times\mathfrak{T}_X(w)\longrightarrow\mathbb{R}$$

defined as

$$X_{u^i},X_{u^j}\longmapsto g_{ij}:=g(X_{u^i},X_{u^j})\in\mathbb{R}.$$

The advantage of this approach is that *we can define a metric g without referring to a special parametrization.* Namely, let $\langle\cdot,\cdot\rangle$ be a positive definite quadratic form on $\mathfrak{T}_X(w)\times\mathfrak{T}_X(w)$. Then we simply set

$$g(\mathscr{X},\mathscr{Y})(w):=\langle\mathscr{X},\mathscr{Y}\rangle(w)$$

for two tangential vectors $\mathscr{X},\mathscr{Y}\in\mathfrak{T}_X(w)$. The form $\langle,\cdot,\cdot\rangle$ could result from projecting the metric of the embedding space to the tangential space of the surface, the former represented by the Euclidean inner product. Then g generates our first fundamental form $I(X)$ from above.

Differential geometry can be formulated completely using this coordinate-free vector field technique, and such an approach takes real account of the *geometrical nature* inherent in our surfaces and manifolds. But eventually it conceals subtle analytical structures of the differential equations underlying the geometric phenomena what would not be detected until we introduce suitable parameter systems.

Chapter 2
Differential equations. Curvatures

In this second chapter we introduce various curvature quantities of surfaces and discuss their geometric invariances. We derive the differential equations of Gauss and Weingarten which will lead us to the mean-curvature-system. Furthermore we introduce a concept of torsions as connection coefficients in the normal space. This is exactly what distinguishes the differential geometry in $\mathbb{R}^3$ from the theory in spaces of higher dimensions.

2.1 The Gauss equations

The connection coefficients of the metric $(g_{ij})_{i,j=1,2}$ are the Christoffel symbols.[1]

Definition 2.1. The *Christoffel symbols* of the immersion $X\colon B \longrightarrow \mathbb{R}^{n+2}$ are

$$\Gamma_{ij}^k := \frac{1}{2}\sum_{k=1}^2 g^{k\ell}(g_{j\ell,u^i} + g_{\ell i,u^j} - g_{ij,u^\ell}) \tag{2.1}$$

for $i,j,k = 1,2,$ where the lower index u^j denotes the partial derivative w.r.t. the parameter u^j.

Their conformal representations can be found in (2.6) below. For the moment we admit arbitrary parametrizations and derive our first set of partial differential equations describing the dynamical behaviour of an immersion in terms of an accompanying frame

$$\{X_u, X_v, N_1, \ldots, N_n\}$$

with a chosen ONF $N = \{N_1, \ldots, N_n\}$. These equations are named after the German mathematician Carl Friedrich Gauss (*1777 in Braunschweig; †1855 in Göttingen).

Theorem 2.1. *(Gauss equations)*
Let the immersion $X\colon B \longrightarrow \mathbb{R}^{n+2}$ together with an ONF N be given. Then there hold

$$X_{u^iu^j} = \sum_{k=1}^2 \Gamma_{ij}^k X_{u^k} + \sum_{\vartheta=1}^n L_{\vartheta,ij} N_\vartheta \tag{2.2}$$

for $i = 1,2$.

[1] The name "connection coefficients" has its origin in the fact that the Γ_{ij}^k determine how to displace parallely a vector on the surface, and thus how some vector of X is related or *connected* to another vector in its neighbourhood; see e.g. Weyl [169], chapter II.

Here we use the abbreviation

$$L_{\omega,ij} := L_{N_\omega,ij}$$

for the coefficients of the second fundamental form w.r.t. the unit normal vector N_ω.

Proof. We evaluate the ansatz

$$X_{u^iu^j} = \sum_{k=1}^{2} a_{ij}^k X_{u^k} + \sum_{\vartheta=1}^{n} b_{\vartheta,ij} N_\vartheta$$

with unknown functions a_{ij}^k and $b_{\vartheta,ij}$. A first multiplication by N_ω yields the coefficients $b_{\vartheta,ij}$ in the form

$$L_{\omega,ij} = -X_{u^i} \cdot N_{\omega,u^j} = X_{u^iu^j} \cdot N_\omega = \sum_{\vartheta=1}^{n} b_{\vartheta,ij} N_\vartheta \cdot N_\omega = b_{\omega,ij} \,.$$

To compute the a_{ij}^k we multiply by X_{u^ℓ} and arrive at

$$X_{u^iu^j} \cdot X_{u^\ell} = \sum_{k=1}^{2} a_{ij}^k X_{u^k} \cdot X_{u^\ell} = \sum_{k=1}^{2} a_{ij}^k g_{k\ell} =: a_{i\ell j} \,. \tag{2.3}$$

Note that $a_{i\ell j} = a_{j\ell i}$ due to $X_{u^iu^j} = X_{u^ju^i}$. We calculate

$$a_{i\ell j} = (X_{u^i} \cdot X_{u^\ell})_{u^j} - X_{u^i} \cdot X_{u^\ell u^j} = g_{i\ell,u^j} - a_{\ell ij},$$

and therefore it holds $g_{i\ell,u^j} = a_{i\ell j} + a_{\ell ij}$. It follows that

$$g_{j\ell,u^i} + g_{\ell i,u^j} - g_{ij,u^\ell} = a_{j\ell i} + a_{\ell ji} + a_{\ell ij} + a_{i\ell j} - a_{ij\ell} - a_{ji\ell} = 2a_{i\ell j} \,,$$

and together with (2.3) we arrive at

$$\sum_{k=1}^{2} a_{ij}^k g_{k\ell} = \frac{1}{2}\,(g_{j\ell,u^i} + g_{\ell i,u^j} - g_{ij,u^\ell}).$$

Rearranging gives

$$a_{ij}^m = \frac{1}{2} \sum_{\ell=1}^{2} g^{m\ell}(g_{j\ell,u^i} + g_{\ell i,u^j} - g_{ij,u^\ell})$$

which already proves the statement. □

With the proof of the Gauss equations we followed the lines of Blaschke and Leichtweiss [15].

Note that the vector $X_{u^iu^j}$ *has no geometric meaning* since the Christoffel symbols Γ_{ij}^k *do not transform* like a $(2,1)$-tensor as the following result shows.

Lemma 2.1. *Let* $u^i(v^\alpha) \in \mathfrak{P}$*. Then there hold*

$$\Gamma_{ij}^k = \sum_{\alpha,\beta,\gamma=1}^2 \Gamma_{\alpha\beta}^\gamma \overline{\Lambda}_\gamma^k \Lambda_i^\alpha \Lambda_j^\beta + \sum_{\alpha=1}^2 \overline{\Lambda}_\alpha^k \Lambda_{i,u^j}^\alpha$$

for $i,j,k = 1,2$ *with the coefficients* Λ_i^α *and* $\overline{\Lambda}_\alpha^i$ *from section 1.7.*

Proof. Recall the transformation formula for g_{ij} and $g^{k\ell}$ from section 1.7. It follows

$$g_{ij,u^k} = \sum_{\alpha,\beta,\gamma=1}^2 \Lambda_i^\alpha \Lambda_j^\beta \Lambda_k^\gamma g_{\alpha\beta,v^\gamma} + \sum_{\alpha,\beta=1}^2 \Lambda_{i,u^k}^\alpha \Lambda_j^\beta g_{\alpha\beta} + \sum_{\alpha,\beta=1}^2 \Lambda_i^\alpha \Lambda_{j,u^k}^\beta g_{\alpha\beta}\,.$$

We compute

$$\begin{aligned}
\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha,\beta,\gamma,\delta,\varepsilon=1}^2 & \overline{\Lambda}_\alpha^k \overline{\Lambda}_\beta^\ell g^{\alpha\beta} \cdot \ldots \\
& \ldots \cdot \left\{ \Lambda_\ell^\gamma \Lambda_i^\delta \Lambda_j^\varepsilon g_{\gamma\delta,u^\varepsilon} + \Lambda_j^\gamma \Lambda_\ell^\delta \Lambda_i^\varepsilon g_{\gamma\delta,u^\varepsilon} - \Lambda_i^\gamma \Lambda_j^\delta \Lambda_\ell^\varepsilon g_{\gamma\delta,u^\varepsilon} \right\} \\
& + \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha,\beta,\gamma,\delta=1}^2 \overline{\Lambda}_\alpha^k \overline{\Lambda}_\beta^\ell g^{\alpha\beta} \left\{ \Lambda_{\ell,u^j}^\gamma \Lambda_i^\delta + \Lambda_{j,u^i}^\gamma \Lambda_\ell^\delta - \Lambda_{i,u^\ell}^\gamma \Lambda_j^\delta \right\} g_{\gamma\delta} \\
& + \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha,\beta,\gamma,\delta=1}^2 \overline{\Lambda}_\alpha^k \overline{\Lambda}_\beta^\ell g^{\alpha\beta} \left\{ \Lambda_\ell^\gamma \Lambda_{i,u^j}^\delta + \Lambda_j^\gamma \Lambda_{\ell,u^i}^\delta - \Lambda_i^\gamma \Lambda_{j,u^\ell}^\delta \right\} g_{\gamma\delta}\,.
\end{aligned}$$

The first row can be rearranged into the form

$$\frac{1}{2} \sum_{\alpha,\beta,\delta,\varepsilon=1}^2 g^{\alpha\beta} \left\{ g_{\beta\delta,u^\varepsilon} + g_{\varepsilon\beta,u^\gamma} - g_{\gamma\varepsilon,u^\delta} \right\} \Lambda_i^\delta \Lambda_j^\varepsilon \overline{\Lambda}_\alpha^k = \sum_{\alpha,\delta,\varepsilon=1}^2 \Gamma_{\delta\varepsilon}^\alpha \Lambda_i^\delta \Lambda_j^\varepsilon \overline{\Lambda}_\alpha^k\,.$$

For the second and the third line it follows

$$\begin{aligned}
& \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta=1}^2 \overline{\Lambda}_\alpha^k \overline{\Lambda}_\beta^\ell g^{\alpha\beta} \cdot \ldots \\
& \quad \ldots \cdot \left\{ \Lambda_{\ell,u^j}^\gamma \Lambda_i^\delta + \Lambda_{i,u^j}^\gamma \Lambda_\ell^\delta - \Lambda_{i,u^\ell}^\gamma \Lambda_j^\delta + \Lambda_{i,u^j}^\gamma \Lambda_\ell^\delta + \Lambda_{i,u^\ell}^\gamma \Lambda_j^\delta - \Lambda_{\ell,u^j}^\gamma \Lambda_i^\delta \right\} g_{\gamma\delta} \\
& = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta=1}^2 \overline{\Lambda}_\alpha^k \overline{\Lambda}_\beta^\ell g^{\alpha\beta} \left\{ \Lambda_{i,u^j}^\gamma \Lambda_\ell^\delta + \Lambda_{i,u^j}^\gamma \Lambda_\ell^\delta \right\} g_{\gamma\delta} = \sum_{\alpha,\beta,\gamma=1}^2 \overline{\Lambda}_\alpha^k \Lambda_{i,u^j}^\gamma g^{\alpha\beta} g_{\gamma\beta} \\
& = \sum_{\alpha=1}^2 \overline{\Lambda}_\alpha^k \Lambda_{i,u^j}^\alpha\,,
\end{aligned}$$

which proves the statement. □

2.2 The mean curvature vector

In the next section we will derive the mean curvature system from the Gauss equations which will be of central interest in later considerations. In preparation for this we start with introducing a normal-dependent mean curvature quantity. Its geometric meaning will be discussed shortly.

Definition 2.2. The *mean curvature* H_N of an immersion $X\colon B \to \mathbb{R}^{n+2}$ *w.r.t. the unit normal vector* $N \in \mathbb{R}^{n+2}$ is defined as

$$H_N := \frac{1}{2}\sum_{i,j=1}^{2} g^{ij}L_{N,ij} = \frac{L_{N,11}g_{22} - 2L_{N,12}g_{12} + L_{N,22}g_{11}}{2W^2}\,. \tag{2.4}$$

Consider now an ONF $N = (N_1, \ldots, N_n)$, and set $H_\sigma := H_{N_\sigma}$.

Definition 2.3. Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then its *mean curvature vector* $H \in \mathbb{R}^n$ is defined as

$$H := \sum_{\sigma=1}^{n} H_\sigma N_\sigma\,. \tag{2.5}$$

Surfaces in $\mathbb{R}^3$ possess, up to orientation, exactly one mean curvature and therefore exactly one mean curvature vector. Its orientations agrees with the orientation of the unit normal vector $N \in \mathbb{R}^3$. In this situation we will emphasize the special nature of the reell number H and the vector $N \in \mathbb{R}^3$.

For the present, H represents vector in $\mathbb{R}^n$ and N a matrix in $\mathbb{R}^{n\times(n+2)}$.

As we will elaborate later, the mean curvature vector H vanishes identically for so-called *minimal surfaces*. Thus the vector H for immersions in $\mathbb{R}^{n+2}$ hardly takes on the role the unit normal vector $N \in \mathbb{R}^3$ claims in case of one codimension. That is one reason why *we rather work with orthonormal normal frames instead of with the mean curvature vector.*

Nevertheless, the vector H possesses interesting analytical and geometric features.

Proposition 2.1. *The curvatures H_N and H are invariant w.r.t. parameter tranformations of class $\mathfrak{P}$. Furthermore the mean curvature vector H does not depend on the choice of the ONF N.*

Proof. Let $N \in \mathbb{R}^{n+2}$ be a unit normal vector. We compute

$$\begin{aligned}\sum_{i,j=1}^{2} g^{ij}L_{N,ij} &= \sum_{i,j=1}^{2}\sum_{\alpha,\beta,\mu,\nu=1}^{2} (g^{\alpha\beta}\overline{\Lambda}{}^i_\alpha\overline{\Lambda}{}^j_\beta)(L_{N,\mu\nu}\Lambda^\mu_i\Lambda^\nu_j) \\ &= \sum_{i,j=1}^{2}\sum_{\alpha,\beta,\mu,\nu=1}^{2} g^{\alpha\beta}L_{N,\mu\nu}\delta^\mu_\alpha\delta^\nu_\beta = \sum_{\mu,\nu=1}^{2} g^{\mu\nu}L_{N,\mu\nu}\end{aligned}$$

from where we already infer the parameter invariance for H_N and H.

Thus we are allowed to introduce conformal parameters. Then, for a frame $\widetilde{N}$, generated by a rotation mapping $\mathbf{R} = (r_{\sigma\omega})_{\sigma,\omega=1,\dots,n}$ as described in the first chapter, we have

$$\begin{aligned}
\sum_{\sigma=1}^{n} \widetilde{H}_\sigma \widetilde{N}_\sigma &= \frac{1}{2W} \sum_{\sigma=1}^{n} (X_{uu} \cdot \widetilde{N}_\sigma + X_{vv} \cdot \widetilde{N}_\sigma) \widetilde{N}_\sigma \\
&= \frac{1}{2W} \sum_{\sigma=1}^{n} \left(\sum_{\omega=1}^{n} \Big\{ X_{uu} \cdot (r_{\sigma\omega} N_\omega) + X_{vv} \cdot (r_{\sigma\omega} N_\omega) \Big\} \right) \left(\sum_{\omega'=1}^{n} r_{\sigma\omega'} N_{\omega'} \right) \\
&= \frac{1}{2W} \sum_{\substack{\omega,\omega'=1 \\ \omega\neq\omega'}}^{n} \left(\sum_{\sigma=1}^{n} r_{\sigma\omega} r_{\sigma\omega'} \right) (X_{uu} \cdot N_\omega + X_{vv} \cdot N_\omega) N_{\omega'} \\
&\quad + \frac{1}{2W} \sum_{\omega=1}^{n} \left(\sum_{\sigma=1}^{n} r_{\sigma\omega} r_{\sigma\omega} \right) (X_{uu} \cdot N_\omega + X_{vv} \cdot N_\omega) N_\omega \\
&= \frac{1}{2W} \sum_{\omega=1}^{n} (X_{uu} \cdot N_\omega + X_{vv} \cdot N_\omega) N_\omega = \sum_{\omega=1}^{n} H_\omega N_\omega \,.
\end{aligned}$$

All statements are proved. □

Let us come back to the normal-dependent curvatures $H_\sigma \equiv H_{N_\sigma}$. Using the foregoing result we can prove they are uniquely defined at one fixed point $w \in B$ if any normal frame N is fixed there.

Proposition 2.2. *Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ together with two ONF N and $\widetilde{N}$ be given. Assume that*

$$N_\sigma(w_0) = \widetilde{N}_\sigma(w_0) \quad \textit{for all } \sigma = 1,\dots,n$$

at some point $w_0 \in B$. Then there hold

$$H_\sigma(w_0) = \widetilde{H}_\sigma(w_0) \quad \textit{for all } \sigma = 1,\dots,n.$$

Proof. Due to the preceding proposition it holds

$$H(w_0) = \sum_{\sigma=1}^{n} H_\sigma(w_0) N_\sigma(w_0) = \sum_{\sigma=1}^{n} \widetilde{H}_\sigma(w_0) \widetilde{N}_\sigma(w_0) = \sum_{\sigma=1}^{n} \widetilde{H}_\sigma(w_0) N_\sigma(w_0).$$

Comparing the components of the basis vectors $N_\sigma(w_0)$ proves the statement. □

2.3 The mean curvature system

From the Gauss equations (2.2) we want to derive an elliptic system for conformally parametrized immersions with prescribed mean curvature vector H.

For this purpose we must rewrite the Christoffel symbols from (2.1) in these special coordinates (the proof is again left to the reader):

$$\begin{aligned} &\Gamma_{11}^1 = \frac{W_u}{2W}\,, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{W_v}{2W}\,, \quad \Gamma_{22}^1 = -\frac{W_u}{2W}\,, \\ &\Gamma_{11}^2 = -\frac{W_v}{2W}\,, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{W_u}{2W}\,, \quad \Gamma_{22}^2 = \frac{W_v}{2W}\,. \end{aligned} \tag{2.6}$$

Theorem 2.2. *(Mean curvature system)*
Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ with mean curvature vector H be given. Then it holds

$$\Delta X := X_{uu} + X_{vv} = 2\sum_{\vartheta=1}^{n} H_\vartheta W N_\vartheta = 2HW \quad \text{in } B. \tag{2.7}$$

Proof. From the Gauss equations (2.2) we infer

$$\Delta X = (\Gamma_{11}^1 + \Gamma_{22}^1)X_u + (\Gamma_{11}^2 + \Gamma_{22}^2)X_v + \sum_{\vartheta=1}^{n} (L_{\vartheta,11} + L_{\vartheta,22})N_\vartheta\,.$$

Note that $\Gamma_{11}^1 + \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 + \Gamma_{22}^2 = 0$ as well as $L_{\vartheta,11} + L_{\vartheta,22} = 2H_\vartheta W$. The statement follows. □

This system (2.7) generalizes the classical mean curvature system

$$\Delta X = 2HWN \quad \text{in } B \tag{2.8}$$

from Hopf [94] valid case $n = 1$ of one codimension with the scalar mean curvature $H \in \mathbb{R}$ and the unit normal vector $N \in \mathbb{R}^3$ of an immersion $X\colon B \to \mathbb{R}^3$.

Definition 2.4. A surfaces is called a *minimal surface* if and only if it holds $H \equiv 0$.

Minimal surfaces are the topic of a countless literature: Courant [38], Nitsche [126], Lawson [112], Giusti [72], Osserman [130], Dierkes et al. [44], Colding and Minicozzi [35], Eschenburg and Jost [52] to enumerate only some few significant distributions and to illustrate the importance of this special surface class in the field of geometric analysis and differential geometry.

2.4 The Gauss curvature

To introduce the Gaussian curvature of an immersion X, we start with the following curvature quantities K_{N_σ} along unit normal vectors N_σ of a given ONF N.

Definition 2.5. The *Gaussian curvature* K of an immersion $X\colon B \to \mathbb{R}^{n+2}$ is

$$K := \sum_{\sigma=1}^{n} K_{N_\sigma}\,, \quad K_{N_\sigma} := \frac{L_{N_\sigma,11}L_{N_\sigma,22} - L_{N_\sigma,12}^2}{g_{11}g_{22} - g_{12}^2}\,. \tag{2.9}$$

Note that due to its dependence on a unit normal vector N, the quantity K_N has no intrinsic nature in contrast to Gaussian curvature K.

Proposition 2.3. *The Gaussian curvature K is invariant w.r.t. parameter transformations of class $\mathfrak{P}$, and it does not depend on the choice of the ONF N.*

Proof. First let N be some unit normal vector. Then the parameter invariance follows from the identities

$$\begin{aligned}
&(N_{u^1}\cdot X_{u^1})(N_{u^2}\cdot X_{u^2})-(N_{u^1}\cdot X_{u^2})^2\\
&\quad=\Big\{(N_{v^1}\cdot X_{v^1})(N_{v^2}\cdot X_{v^2})-(N_{v^1}\cdot X_{v^2})^2\Big\}\Big\{\Lambda_1^1\Lambda_2^2-\Lambda_1^2\Lambda_2^1\Big\}^2,\\
&(X_{u^1}\cdot X_{u^1})(X_{u^2}\cdot X_{u^2})-(X_{u^1}\cdot X_{u^2})^2\\
&\quad=\Big\{(X_{v^1}\cdot X_{v^1})(X_{v^2}\cdot X_{v^2})-(X_{v^1}\cdot X_{v^2})^2\Big\}\Big\{\Lambda_1^1\Lambda_2^2-\Lambda_1^2\Lambda_2^1\Big\}^2.
\end{aligned}$$

To complete this first part of the proof, the reader should check a analogous transformation formula for the determinant $g_{11}g_{22}-g_{12}^2$! Let us now come to the invariance w.r.t. to orthonormal normal frames. We have

$$\widetilde{K}_\sigma W^2=\widetilde{L}_{\sigma,11}\widetilde{L}_{\sigma,22}-\widetilde{L}_{\sigma,12}^2=(X_{uu}\cdot\widetilde{N}_\sigma)(X_{vv}\cdot\widetilde{N}_\sigma)-(X_{uv}\cdot\widetilde{N}_\sigma)^2,$$

and the right hand side of this equation can be written in the following form

$$\begin{aligned}
&\left(\sum_{\omega=1}^n r_{\sigma\omega}L_{\omega,11}\right)\left(\sum_{\omega'=1}^n r_{\sigma\omega'}L_{\omega',22}\right)-\left(\sum_{\omega=1}^n r_{\sigma\omega}L_{\omega,12}\right)^2\\
&\quad=\sum_{\omega,\omega'=1}^n r_{\sigma\omega}r_{\sigma\omega'}\big(L_{\omega,11}L_{\omega',22}-L_{\omega,12}L_{\omega',12}\big)\\
&\quad=\sum_{\omega=1}^n r_{\sigma\omega}^2\big(L_{\omega,11}L_{\omega,22}-L_{\omega,12}L_{\omega,12}\big)\\
&\qquad+\sum_{\substack{\omega,\omega'=1\\ \omega\neq\omega'}}^n r_{\sigma\omega}r_{\sigma\omega'}\big(L_{\omega,11}L_{\omega',22}-L_{\omega,12}L_{\omega',12}\big).
\end{aligned}$$

Taking the results of section 1.4 into account we have

$$\begin{aligned}
\sum_{\sigma=1}^n\sum_{\omega=1}^n r_{\sigma\omega}^2\big(L_{\omega,11}L_{\omega,22}-L_{\omega,12}^2\big)&=\sum_{\omega=1}^n\left(\sum_{\sigma=1}^n r_{\sigma\omega}^2\right)\big(L_{\omega,11}L_{\omega,22}-L_{\omega,12}^2\big)\\
&=\sum_{\omega=1}^n\big(L_{\omega,11}L_{\omega,22}-L_{\omega,12}^2\big).
\end{aligned}$$

Analogously it follows

$$\sum_{\sigma=1}^{n}\sum_{\substack{\omega,\omega'=1\\ \omega\neq\omega'}}^{n} r_{\sigma\omega}r_{\sigma\omega'}\big(L_{\omega,11}L_{\omega',22}-L_{\omega,12}L_{\omega',12}\big)=0.$$

Thus we conclude

$$\widetilde{K}W^2=\sum_{\sigma=1}^{n}\widetilde{K}_\sigma W^2=\sum_{\omega=1}^{n}\big(L_{\omega,11}L_{\omega,22}-L_{\omega,12}^2\big)=\sum_{\omega=1}^{n}K_\omega W^2=KW^2$$

proving the proposition. □

The Gaussian curvature K belongs to the inner geometry of an immersion and neither depends on the parametrization nor its embedding in space.

In the next chapter we will prove that K can be expressed using the surface's first fundamental form and its first and second derivatives exclusively. This is the contents of the well-known *theorema egregium.*

No matter the geometric meaning of the normal-dependent curvatures K_σ actually is, we next want to show

Proposition 2.4. *Given the immersion $X\colon B\longrightarrow\mathbb{R}^{n+2}$ together with two ONF N and $\widetilde{N}$. Suppose that*

$$N_\sigma(w_0)=\widetilde{N}_\sigma(w_0)\quad\textit{for all }\sigma=1,\ldots,n$$

at some fixed point $w_0\in\mathring{B}$. Then there hold

$$K_\sigma(w_0)=\widetilde{K}_\sigma(w_0)\quad\textit{for all }\sigma=1,\ldots,n.$$

Proof. We calculate

$$\begin{aligned}
&L_{\sigma,11}(w_0)L_{\sigma,22}(w_0)-L_{\sigma,12}(w_0)^2\\
&\quad=\big[X_{uu}\cdot N_\sigma(w_0)\big]\big[X_{vv}\cdot N_\sigma(w_0)\big]-\big[X_{uv}\cdot N_\sigma(w_0)\big]^2\\
&\quad=\big[X_{uu}\cdot \widetilde{N}_\sigma(w_0)\big]\big[X_{vv}\cdot \widetilde{N}_\sigma(w_0)\big]-\big[X_{uv}\cdot \widetilde{N}_\sigma(w_0)\big]^2\\
&\quad=\widetilde{L}_{\sigma,11}(w_0)\widetilde{L}_{\sigma,22}(w_0)-\widetilde{L}_{\sigma,12}(w_0)^2\,.
\end{aligned}$$

The statement is proved. □

2.5 The normal mean and the normal Gauss curvature

Beside the three fundamental forms introduced in section 1.5 there is a varity of further quadratic forms reflecting the geometry of higher-codimensional surfaces and leading to new curvature concepts.

Conveniently one defines the following quantities.

Definition 2.6. Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ be given together with an ONF N. Let furthermore

$$L_{\sigma\vartheta,11} := \frac{2}{W}\begin{vmatrix} L_{\sigma,11} & L_{\sigma,12} \\ L_{\vartheta,11} & L_{\vartheta,12}\end{vmatrix}, \quad L_{\sigma\vartheta,22} := \frac{2}{W}\begin{vmatrix} L_{\sigma,12} & L_{\sigma,22} \\ L_{\vartheta,12} & L_{\vartheta,22}\end{vmatrix},$$

$$L_{\sigma\vartheta,12} := \frac{1}{W}\begin{vmatrix} L_{\sigma,11} & L_{\sigma,22} \\ L_{\vartheta,11} & L_{\vartheta,22}\end{vmatrix}$$

for $\sigma,\vartheta=1,\ldots,n$. Then the *normal mean curvature matrix* and the *normal Gaussian curvature matrix* of X w.r.t. N are defined as

$$\mathbf{H} := (H_{\sigma\vartheta})_{\sigma,\vartheta=1,\ldots,n} \quad \text{with} \quad H_{\sigma\vartheta} := \sum_{i,j=1}^{2} g^{ij}L_{\sigma\vartheta,ij}\,,$$

$$\mathbf{K} := (K_{\sigma\vartheta})_{\sigma,\vartheta=1,\ldots,n} \quad \text{with} \quad K_{\sigma\vartheta} := \frac{L_{\sigma\vartheta,11}L_{\sigma\vartheta,22}-L_{\sigma\vartheta,12}^2}{W^2}$$

for $\sigma,\vartheta=1,\ldots,n$.

The curvatures $H_{\sigma\vartheta}$ and $K_{\sigma\vartheta}$ are *sectional curvatures* in the following sense.

Proposition 2.5. *For all $\sigma,\vartheta=1,\ldots,n$ there hold the following statements.*

1. *The curvatures $H_{\sigma\vartheta}$ and $K_{\sigma\vartheta}$ are invariant w.r.t. parameter transformations of class $\mathfrak{P}$.*
2. *The quadratic forms $L_{\sigma\vartheta,ij}$, $i,j=1,2$, and therefore the curvatures $H_{\sigma\vartheta}$ and $K_{\sigma\vartheta}$ do not depend on the choice of an orthonormal basis of* Span$\{N_\sigma,N_\vartheta\}$.

In other words, **H** *and* **K** *are geometric quantities.*

Proof. The parameter invariance can be proved using the methods from our proof of Proposition 2.3. Thus we only consider the second statement. Let us start with two unit normal vectors N_σ and N_ϑ with $\sigma\neq\vartheta$, taken from some ONF N, and evaluate the $SO(2)$-action on this frame, i.e. let

$$\widetilde{N}_\sigma := \cos\varphi\,N_\sigma+\sin\varphi\,N_\vartheta\,, \quad \widetilde{N}_\vartheta := -\sin\varphi\,N_\sigma+\cos\varphi\,N_\vartheta$$

be another orthonormal basis of Span$\{N_\sigma,N_\vartheta\}$.

Then we exemplarily calculate

$$\begin{aligned}\frac{W}{2}\,\widetilde{L}_{\sigma\vartheta,11} &= \widetilde{L}_{\sigma,11}\widetilde{L}_{\vartheta,12}-\widetilde{L}_{\sigma,12}\widetilde{L}_{\vartheta,11}\\ &= -\sin\varphi\cos\varphi\,L_{\sigma,11}L_{\sigma,12}+\cos^2\varphi\,L_{\sigma,11}L_{\vartheta,12}+\sin^2\varphi\,L_{\sigma,12}L_{\vartheta,11}\\ &\quad +\sin\varphi\cos\varphi\,L_{\vartheta,11}L_{\vartheta,12}+\sin\varphi\cos\varphi\,L_{\sigma,11}L_{\sigma,12}-\cos^2\varphi\,L_{\sigma,12}L_{\vartheta,11}\\ &\quad +\sin^2\varphi\,L_{\sigma,12}L_{\vartheta,12}-\sin\varphi\cos\varphi\,L_{\vartheta,11}L_{\vartheta,12}\\ &= L_{\sigma,11}L_{\vartheta,12}-L_{\sigma,12}L_{\vartheta,11} = \frac{W}{2}\,L_{\sigma\vartheta,11}\,.\end{aligned}$$

Analogously we prove the invariance of $L_{\sigma\vartheta,12}$ and $L_{\sigma\vartheta,22}$. □

We will come back the curvatures quantities $H_{\sigma\vartheta}$ in section 3.7 again.

2.6 The Weingarten equations

New aspects in the analysis of surfaces in spaces of higher codimensions are mainly manifested in the *connection coefficients of the normal space* or, as we prefer to say, in the *torsion coefficients.*

Definition 2.7. The *torsion coefficients* of an ONF N are defined as

$$T_{\sigma,i}^{\vartheta} := N_{\sigma,u^i}\cdot N_\vartheta\,, \quad T_{\sigma,i}^{\vartheta} = -T_{\vartheta,i}^{\sigma}\,, \tag{2.10}$$

for $i=1,2$ and $\sigma,\vartheta=1,\ldots,n$.

Most of our definitions and identities can be found in various textbooks on differential geometry, for example Chen [27], Brauner [18], or do Carmo [22]. The terminology “torsion” even essentially goes back to Weyl [168]: *Aus einem normalen Vektor* $\mathbf{n}$ *in P entsteht ein Vektor* $\mathbf{n}'+d\mathbf{t}$ *(*$\mathbf{n}'$ *normal,* $d\mathbf{t}$ *tangential). Die infinitesimale lineare Abbildung* $\mathbf{n}\to\mathbf{n}'$ *von* $\mathfrak{N}_P$ *auf* $\mathfrak{N}_{P'}$ *ist die Torsion.*[2]

In chapters 7 and 8 we will elaborately study special normal frames with special torsions. For the moment we concentrate on the basic differential equations of surface theory.

Theorem 2.3. *(Weingarten equations)*
Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ *with an ONF* N *be given. Then there hold*

$$N_{\sigma,u^i} = -\sum_{j,k=1}^{2} L_{\sigma,ij}g^{jk}X_{u^k}+\sum_{\vartheta=1}^{n} T_{\sigma,i}^{\vartheta}N_\vartheta \tag{2.11}$$

for $i=1,2$ *and* $\sigma=1,\ldots,n$.

[2] From a normal vector $\mathbf{n}$ in P there arises a vector $\mathbf{n}'+d\mathbf{t}$ ($\mathbf{n}'$ normal, $d\mathbf{t}$ tangential). The infinitesimal linear mapping $\mathbf{n}\to\mathbf{n}'$ of $\mathfrak{N}_P$ to $\mathfrak{N}_{P'}$ is the torsion.

Proof. With unknown functions $a_{\sigma,i}$ and $b_{\sigma,i}^{\vartheta}$ we evaluate the ansatz

$$N_{\sigma,u^i} = \sum_{k=1}^{2} a_{\sigma,i}^{k} X_{u^k} + \sum_{\vartheta=1}^{n} b_{\sigma,i}^{\vartheta} N_{\vartheta} \,.$$

Multiplication by X_{u^ℓ} gives

$$-L_{\sigma,i\ell} = N_{\sigma,u^i} \cdot X_{u^\ell} = \sum_{k=1}^{2} a_{\sigma,i}^{k} X_{u^k} \cdot X_{u^\ell} = \sum_{k=1}^{2} a_{\sigma,i}^{k} g_{k\ell} \,,$$

and rearranging yields

$$a_{\sigma,i}^{m} = -\sum_{\ell=1}^{2} L_{\sigma,i\ell} g^{\ell m} \,.$$

A further multiplication by N_ω shows

$$T_{\sigma,i}^{\omega} = N_{\sigma,u^i} \cdot N_\omega = \sum_{\vartheta=1}^{n} b_{\sigma,i}^{\vartheta} N_\vartheta \cdot N_\omega = \sum_{\vartheta=1}^{n} b_{\sigma,i}^{\vartheta} \delta_{\vartheta\omega} = b_{\sigma,i}^{\omega} \,,$$

which proves the statement. □

The system (2.11) generalizes the classical Weingarten equations

$$N_{u^i} = -\sum_{j,k=1}^{2} L_{ij} g^{jk} X_{u^k} \,, \quad i = 1,2,$$

for the unit normal vector N of a surface $X\colon B \longrightarrow \mathbb{R}^{n+2}$ in the case of one codimension $n=1$, found by the German mathematician Julius Weingarten (*1836 in Berlin; †1910 in Freiburg). All torsion coefficients vanish identically here.

There are also immersions living in higher dimensional spaces and admitting orthonormal normal frames which are *globally free of torsion.* We will discuss their properties extensively in later sections.

Finally we want to mention that the $T_{\sigma,i}^{\vartheta}$ transform like a $(1,0)$-tensor. The proof is left to the reader.

Proposition 2.6. *Let* $u^i(v^\alpha) \in \mathfrak{P}$. *Then there hold*

$$T_{\sigma,i}^{\vartheta} = \sum_{\mu=1}^{2} T_{\sigma,\mu}^{\vartheta} \Lambda_i^{\mu}$$

for $i = 1,2$ *and* $\sigma, \vartheta = 1,\ldots,n$.

2.7 Weingarten forms. Principal curvatures

In this section we want to introduce the so-called Weingarten forms to define algebraically the principal curvatures of an immersion.

Definition 2.8. The *Weingarten forms* $(L_{N,i}^k)_{i,k=1,2}$ of an immersion $X\colon B \to \mathbb{R}^{n+2}$ w.r.t. some unit normal vector $N \in \mathbb{R}^{n+2}$ are defined by

$$L_{N,i}^k := L_{N,ij}g^{jk} \quad \text{for } i,k = 1,2.$$

Note that due to the definition of the curvature quantities H_N from (2.4) and the Gauss curvatures K_N from (2.9) along the unit normal vector N we infer

$$\begin{aligned} L_{N,1}^1 + L_{N,2}^2 &= \sum_{j=1}^{2} L_{N,1j}g^{j1} + \sum_{j=1}^{2} L_{N,2j}g^{j2} \\ &= \sum_{i,j=1}^{2} L_{N,ij}g^{ji} = 2H_N \end{aligned}$$

as well as

$$\begin{aligned} L_{N,1}^1 L_{N,2}^2 - (L_{N,1}^2) &= \sum_{i,j=1}^{2} L_{N,1i}L_{N,2j}g^{i1}g^{j2} - \sum_{i,j=1}^{2} L_{N,1i}g^{i2}L_{N,2j}g^{j1} \\ &= (L_{N,11}L_{N,22} - L_{N,12}^2)(g^{11}g^{22} - g^{12}g^{21}) = K_N\,. \end{aligned}$$

The principal curvatures are now defined for each particular unit normal vector N as follows.

Definition 2.9. The *principal curvatures* $\kappa_{N,1}$ and $\kappa_{N,2}$ of an immersion $X\colon B \to \mathbb{R}^{n+2}$ w.r.t. the unit normal vector $N \in \mathbb{R}^{n+2}$ are defined as the roots of the quadratic eigenvalue equation

$$\det(L_{N,i}^j - \lambda_N \delta_i^j)_{i,j=1,2} = 0.$$

Note that $H_N^2 - K_N \geq 0$ such that the corresponding eigenvalues follow from

$$\kappa_{N,1} = H_N - \sqrt{H_N^2 - K_N}\,, \quad \kappa_{N,2} = H_N + \sqrt{H_N^2 - K_N}\,,$$

or equivalently

$$H_N = \frac{\kappa_{N,1} + \kappa_{N,2}}{2}\,, \quad K_N = \kappa_{N,1}\kappa_{N,2}\,.$$

Thus all the classical methods from the theory of surfaces in $\mathbb{R}^3$ apply for each single H_N and K_N.

2.8 Application I: Geometry of the Gaussian curvature

Gauss himself approached to the curvature quantity K for surfaces in $\mathbb{R}^3$ from a purely geometrical point of view (we quote from Gauss [70], pp. 10–11):

> . . . to each part of a curved surface inclosed within definite limits we assign a *total* or *integral curvature,* which is represented by the area of the figure on the sphere corresponding to it. From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the *measure of curvature.* The latter refers to a *point* of the surface and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself; and hence it denotes the ratio of the infinitely small areas which correspond to one another on the curved surface and on the sphere.

Or using modern mathematical notation:

$$K = \frac{|N_u \times N_v|}{|X_u \times X_v|} \in S^2 := \big\{ Z \in \mathbb{R}^3 : |Z| = 1 \big\} \subset \mathbb{R}^3$$

with the unit normal vector $N \subset \mathbb{R}^3$ of the immersion $X : B \to \mathbb{R}^3$ which maps into the sphere $S^2 \subset \mathbb{R}^3$.

Let us now consider again the general situation of immersions $X : B \to \mathbb{R}^{n+2}$. Suppose the surface admits an ONF N which is *free of torsion,* i.e.

$$T_{\sigma,i}^{\vartheta} \equiv 0 \quad \text{for all } i = 1,2,\ \sigma, \vartheta = 1, \ldots, n.$$

Existence of torsion free orthonormal normal frames N is strongly coupled with a quantity representing the *curvature of the normal bundle.* We will introduce this curvature tensor in the next chapter, and in chapters 8 and 9 we will attack the problem of constructing smooth torsion free ONF.

Let $(u,v) \in B$ be conformal parameters. The *area of the spherical image* of some unit normal vector N_σ can be computed from

$$\text{Area}\,(N_\sigma)^2 = N_{\sigma,u}^2 N_{\sigma,v}^2 - (N_{\sigma,u} \cdot N_{\sigma,v})^2 .$$

Inserting the Weingarten equations with zero torsion coefficients yields

$$\begin{aligned}
\text{Area}\,(N_\sigma)^2 &= \frac{1}{W^2}\,(L_{\sigma,11}^2 + L_{\sigma,12}^2)(L_{\sigma,12}^2 + L_{\sigma,22}^2) - \frac{1}{W^2}\,(L_{\sigma,11}L_{\sigma,12} + L_{\sigma,12}L_{\sigma,22})^2 \\
&= \frac{1}{W^2}\,(L_{\sigma,11}^2 L_{\sigma,22}^2 + L_{\sigma,12}^4 - 2L_{\sigma,11}L_{\sigma,22}L_{\sigma,12}^2) \\
&= \frac{1}{W^2}\,(L_{\sigma,11}L_{\sigma,22} - L_{\sigma,12}^2)^2 .
\end{aligned}$$

Proposition 2.7. *Let the immersion $X\colon\, B \longrightarrow \mathbb{R}^{n+2}$ together with an ONF N be given. Suppose that N is free of torsion. Then it holds*

$$\mathrm{Area}\,(N_\sigma) = |K_{N_\sigma}|W^2$$

for each $N_\sigma \in N$.

Of course, this is also true for immersions $X\colon\, B \to \mathbb{R}^3$, but in general it fails in case of higher codimensions.

2.9 Application II: Surfaces with parallel mean curvature vector

Using (2.11) together with the skew-symmetry of the torsion coefficients we want to compute the normal components of the derivative of the mean curvature vector (denoted by the superscrip $\perp$):

$$\begin{aligned} H_u^\perp &= \sum_{\sigma=1}^n (H_{\sigma,u}N_\sigma + H_\sigma N_{\sigma,u})^\perp = \sum_{\sigma=1}^n H_{\sigma,u}N_\sigma + \sum_{\sigma,\vartheta=1}^n H_\sigma T_{\sigma,1}^\vartheta N_\vartheta \\ &= \sum_{\sigma=1}^n \left\{ H_{\sigma,u} - \sum_{\vartheta=1}^n H_\vartheta T_{\sigma,1}^\vartheta \right\} N_\sigma \end{aligned}$$

and analogously

$$H_v^\perp = \sum_{\sigma=1}^n \left\{ H_{\sigma,v} - \sum_{\vartheta=1}^n H_\vartheta T_{\sigma,2}^\vartheta \right\} N_\sigma\,.$$

Definition 2.10. The mean curvature vector H is called *parallel in the normal bundle* if and only if

$$H_u^\perp \equiv 0 \quad \text{and} \quad H_v^\perp \equiv 0,$$

or equivalently

$$H_{\sigma,u^i} = \sum_{\vartheta=1}^n H_\vartheta T_{\sigma,i}^\vartheta \quad \text{for all } i=1,2,\ \sigma = 1,\ldots,n,$$

with respect to an arbitrary ONF N.

From this definition we immediately read elementary examples for surfaces with mean curvature vector parallel in the normal bundle: Namely,

- minimal surfaces with the property $H \equiv 0$;
- surfaces which admit a torsion free ONF N satisfying $T_{\sigma,i}^\vartheta \equiv 0$.

Let now $H_\sigma \neq 0$, and suppose that the mean curvature vector H is parallel in the normal bundle.

Multiplication of the first order differential equations from the foregoing definition by H_σ gives

$$H_\sigma H_{\sigma,u^i} - \sum_{\vartheta=1}^{n} H_\sigma H_\vartheta T_{\sigma,i}^{\vartheta} = 0$$

for all $\sigma = 1,\ldots,n$. Summation over σ shows

$$\frac{1}{2}\frac{\partial}{\partial u^i}|H|^2 = \sum_{\sigma=1}^{n} H_\sigma H_{\sigma,u^i} = \sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n} H_\sigma H_\vartheta T_{\sigma,i}^{\vartheta} = 0$$

where the right hand side vanishes automatically due to the skew-symmetry of the torsion coefficients. *Thus we conclude*

$$|H|^2 = H_1^2 + H_2^2 + \ldots + H_n^2 = \text{const}$$

if and only if H is parallel in the normal bundle.

In fact, skipping geometrical details, we have proved the following classification result due to Chen from [26].

Theorem 2.4. *If $H \neq 0$ is parallel in the normal bundle, then either the immersion is a minimal surface of a hypersphere of $\mathbb{R}^{n+2}$, or it has flat normal bundle.*

As it turns out in the course of our following considerations, *flat normal bundle* is just equivalent to the existence of a torsion free ONF N. But it is common practise in differential geometry to speak of a flat normal bundle as a geometric property invariantly linked with the surface instead of stressing special properties of moving normal frames.

Further Remarks

Surfaces with mean curvature vector parallel in the normal bundle in $\mathbb{R}^{n+2}$ can be compared with surface of constant mean curvature in $\mathbb{R}^3$. Since the theory of this class of immersions has its own very interesting history, we want to refer some cornerstones of this branch of differential geometry.

We abbreviatorily denote by $\mathscr{M} \subset \mathbb{R}^3$ a twodimensional surface as a set in space.[3] Then we start with two results which go already back to Heinrich Liebmann (*1874 in Straßburg; †1939 in München) from 1899, see Liebmann [113], [114].

Theorem 2.5. *1. Let $\mathscr{M} \subset \mathbb{R}^3$ be given with constant Gauss curvature $K > 0$. Then it holds $K > 0$, and the surface is a round sphere of radius $\frac{1}{\sqrt{K}}$.*

2. Let $\mathscr{M} \subset \mathbb{R}^3$ be given with Gauss curvature $K > 0$ and constant scalar mean curvature H. Then the surface is a round sphere of radius $\frac{1}{|H|}$.

[3] In fact, this point of view requires some *theory of manifolds* which we do not discuss here.

Heinz Hopf (*1894 near Breslau; †1971 in Zollikon) in [94] succeeded *without assuming the convexity* $K > 0$.

Theorem 2.6. *Let the surface* $\mathscr{M} \subset \mathbb{R}^3$ *be given with constant scalar mean curvature. Then* $\mathscr{M}$ *is a sphere.*

The question arose whether beside the sphere there is a further compact immersion without boundary with constant mean curvature in $\mathbb{R}^3$. This problem is subject to the so-called *Hopf conjecture.* In 1986, the American mathematician Henry Wente in [167] proved the existence of such a constant mean curvature immersion, now of the type of the torus.

Theorem 2.7. *There is a conformal immersion of* $\mathbb{R}^2$ *into* $\mathbb{R}^3$ *with constant scalar mean curvature* $H \neq 0$ *which is doubly-periodic with respect to a rectangle in* $\mathbb{R}^2$.

A simplified but detailed proof of Wente's result can be found in Abresch [1]. We also want to refer to Glaeser and Polthier [73] for excellent numerical presentations of compact constant mean curvature surfaces as well as references to mathematical literature on this subject before Wente, see e.g. Dobriner [45].

There is an endless list of contemporary studies on constant mean curvature surfaces. The reader finds various excellent contributions in the works of U. Abresch, B. Ammann, C. Gerhardt, K. Grosse-Brauckmann, F. Helein, J. Isenberg, H. Karcher, M. Kilian, K. Kenmotsu, N. Kapouleas, R. Lopez, R. Kusner, F. Martin, W.H. Meeks, F. Pedit, K. Polthier, N. Schmidt, J. Sullivan, M. Weber, H. Wente etc.

In 1972, David Hoffman in [92] considered the embedding problem for compact surfaces with parallel mean curvature vector in four-dimensional Euclidean space.

Theorem 2.8. *1. Let* $\mathscr{M} \subset \mathbb{R}^4$ *be given, and suppose that its mean curvature vector* $H \in \mathbb{R}^4$ *is parallel in the normal bundle. Then the surface is a round sphere of radius* $\frac{1}{|H|}$.

2. Let $\mathscr{M} \subset \mathbb{R}^4$ *be given, and suppose that its Gaussian curvature* K *does not change sign, and that its mean curvature vector* $H \in \mathbb{R}^4$ *is parallel in the normal bundle. Then* $\mathscr{M}$ *is a minimal surface, i.e.* $H \equiv 0$, *a sphere, a right circular cylinder, or a product of circles* $S^1(r) \times S^1(\rho)$, *where* $|H| = \frac{1}{2}\,(\frac{1}{r^2} + \frac{1}{\rho^2})^{\frac{1}{2}}$.

3. A piece of an immersed surface $\mathscr{M} \subset \mathbb{R}^4$, *satisfying the conditions of point 2 as well as* $H \neq 0$, *is either a piece of the round sphere, or it is flat with* $K = 0$ *for its Gaussian curvature.*

Finally we want to quote the following generalization of Liebmann's theorem to the case of arbitrary codimension from 1985 due to Enmoto [51].

Theorem 2.9. *Let* $\mathscr{M} \subset \mathbb{R}^{n+2}$ *be a regular, closed, simply connected, and compact surface with Gauss curvature* $K > 0$. *Suppose that* $|H| = const$, *and that its normal bundle is flat. Then the surface is a round sphere in a three-dimensional affine space* $\mathbb{R}^3 \subset \mathbb{R}^{n+2}$.

Chapter 3
Integrability conditions

In this third chapter we derive the integrability conditions of Codazzi-Mainardi, Ricci and the theorema egregium from the equations of Gauss and Weingarten. In particular, we introduce the Riemannian curvature tensor and the curvature tensor of the normal bundle. We conclude this chapter with a general version of the fundamental theorem of surface theory.

3.1 Integrability conditions

To emphasize the special role certain partial derivative eventually are playing in our analysis, we introduce the notation

$$\partial_{u^i} X := X_{u^i}\,, \quad i = 1,2,$$

denoting the partial differentiation of X w.r.t. u^i.

Then in view of the regularity assumptions $X \in C^4(B,\mathbb{R}^n)$ and $N_\sigma \in C^3(B,\mathbb{R}^n)$ there hold necessarily

$$\partial_{uv} X_{u^i} - \partial_{vu} X_{u^i} \equiv 0, \quad \partial_{uv} N_\sigma - \partial_{vu} N_\sigma \equiv 0 \tag{3.1}$$

for all $i = 1,2$ and all $\sigma = 1,\ldots,n$ due to an elementary theorem of H.A. Schwarz.

Evaluating these identities, taking the equations of Gauss (2.2) and Weingarten (2.11) into account, gives the following three *integrability conditions.*

- Integrability conditions w.r.t. the derivatives of the surface vector X

$$\partial_{uv} X_{u^i} - \partial_{vu} X_{u^i} \equiv 0$$

$$(\partial_{uv} X_{u^i} - \partial_{vu} X_{u^i})^{\text{norm}} \equiv 0 \qquad\qquad (\partial_{uv} X_{u^i} - \partial_{vu} X_{u^i})^{\text{tang}} \equiv 0$$

Codazzi-Mainardi equations (Section 3.2) — theorema egregium (Sections 3.3, 3.5)

To be precise, the Gauss equations (2.2) can be written in the form

$$\partial_u X_{u^i} = \Gamma^1_{i1} X_u + \Gamma^2_{i1} X_v + \sum_{\sigma=1}^{n} L_{\sigma,i1} N_\sigma \,, \quad \partial_v X_{u^i} = \Gamma^1_{i2} X_u + \Gamma^2_{i2} X_v + \sum_{\sigma=1}^{n} L_{\sigma,i2} N_\sigma$$

for $i = 1,2$. Differentiation, while taking the Weingarten equations and the Gauss equations into account, yields

$$\begin{aligned} \partial_{uv} X_{u^i} = & \left\{ \Gamma^1_{i1,v} + \Gamma^1_{i1}\Gamma^1_{12} + \Gamma^2_{i1}\Gamma^1_{22} - \sum_{\omega=1}^{n} L_{\omega,i1} L_{\omega,2\ell} g^{\ell 1} \right\} X_u \\ & + \left\{ \Gamma^2_{i1,v} + \Gamma^1_{i1}\Gamma^2_{12} + \Gamma^2_{i1}\Gamma^2_{22} - \sum_{\omega=1}^{n} L_{\omega,i1} L_{\omega,2\ell} g^{\ell 2} \right\} X_v \\ & + \sum_{\sigma=1}^{n} \left\{ L_{\sigma,i1,v} + \Gamma^1_{i1} L_{\sigma,12} + \Gamma^2_{i1} L_{\sigma,22} + \sum_{\omega=1}^{n} L_{\omega,i1} T^\sigma_{\omega,2} \right\} N_\sigma \,, \end{aligned} \tag{3.2}$$

as well as

$$\begin{aligned} \partial_{vu} X_{u^i} = & \left\{ \Gamma^1_{i2,u} + \Gamma^1_{i2}\Gamma^1_{11} + \Gamma^2_{i2}\Gamma^1_{12} - \sum_{\omega=1}^{n} L_{\omega,i2} L_{\omega,1\ell} g^{\ell 1} \right\} X_u \\ & + \left\{ \Gamma^2_{i2,u} + \Gamma^1_{i2}\Gamma^2_{11} + \Gamma^2_{i2}\Gamma^2_{12} - \sum_{\omega=1}^{n} L_{\omega,i2} L_{\omega,1\ell} g^{\ell 2} \right\} X_v \\ & + \sum_{\sigma=1}^{n} \left\{ L_{\sigma,i2,u} + \Gamma^1_{i2} L_{\sigma,11} + \Gamma^2_{i2} L_{\sigma,12} + \sum_{\omega=1}^{n} L_{\omega,i2} T^\sigma_{\omega,1} \right\} N_\sigma \,. \end{aligned} \tag{3.3}$$

From these identities we will derive the *Codazzi-Mainardi equations* and the *theorema egregium.*

- Integrability conditions w.r.t. the derivatives of unit normal vectors $N_\sigma \in \mathbb{R}^{n+2}$

$$\partial_{uv} N_\sigma - \partial_{vu} N_\sigma \equiv 0$$

↙ ↘

$$(\partial_{uv} N_\sigma - \partial_{vu} N_\sigma)^{\text{tang}} \equiv 0 \qquad\qquad (\partial_{uv} N_\sigma - \partial_{vu} N_\sigma)^{\text{norm}} \equiv 0$$

↓ ↓

Codazzi-Mainardi equations (Section 3.2) Ricci equations (Sections 3.6, 3.7)

Now we evaluate the integrability conditions for the unit normal vectors. To this end, we first compute

$$\begin{aligned}
\partial_{uv}N_\sigma &= -\sum_{j,k=1}^{2} L_{\sigma,1j,v}g^{jk}X_{u^k} - \sum_{j,k=1}^{2} L_{\sigma,1j}g^{jk}_{\ ,v}X_{u^k} - \sum_{j,k=1}^{2} L_{\sigma,1j}g^{jk}X_{u^kv} \\
&\quad + \sum_{\omega=1}^{n} T^{\omega}_{\sigma,1,v}N_\omega + \sum_{\omega=1}^{n} T^{\omega}_{\sigma,1}N_{\omega,v} \\
&= -\sum_{k=1}^{2}\left\{\sum_{j=1}^{2} L_{\sigma,1j,v}g^{jk} + \sum_{j=1}^{2} L_{\sigma,1j}g^{jk}_{\ ,v} + \sum_{j,m=1}^{2} L_{\sigma,1j}g^{jm}\Gamma^k_{m2} + \ldots\right. \\
&\qquad\qquad \left.\ldots + \sum_{j=1}^{2}\sum_{\vartheta=1}^{n} T^{\vartheta}_{\sigma,1}L_{\vartheta,2j}g^{jk}\right\} X_{u^k} \\
&\quad - \sum_{\omega=1}^{n}\left\{\sum_{j,k=1}^{2} L_{\sigma,1j}g^{jk}L_{\omega,k2} - T^{\omega}_{\sigma,1,v} - \sum_{\vartheta=1}^{n} T^{\vartheta}_{\sigma,1}T^{\omega}_{\vartheta,2}\right\} N_\omega
\end{aligned} \tag{3.4}$$

and analogously

$$\begin{aligned}
\partial_{vu}N_\sigma &= -\sum_{k=1}^{2}\left\{\sum_{j=1}^{2} L_{\sigma,2j,u}g^{jk} + L_{\sigma,2j}g^{jk}_{\ ,u} + \sum_{j=1}^{2}\sum_{j,m=1}^{2} L_{\sigma,2j}g^{jm}\Gamma^k_{m1} + \ldots\right. \\
&\qquad\qquad \left.\ldots + \sum_{j=1}^{2}\sum_{\vartheta=1}^{n} T^{\vartheta}_{\sigma,2}L_{\vartheta,1j}g^{jk}\right\} X_{u^k} \\
&\quad - \sum_{\omega=1}^{n}\left\{\sum_{j,k=1}^{2} L_{\sigma,2j}g^{jk}L_{\omega,k1} - T^{\omega}_{\sigma,2,u} - \sum_{\vartheta=1}^{n} T^{\vartheta}_{\sigma,2}T^{\omega}_{\vartheta,1}\right\} N_\omega
\end{aligned} \tag{3.5}$$

using the Gauss and the Weingarten equations.

Comparing tangential and normal parts of these identities yields the so-called *Codazzi-Mainardi equations* resp. *Ricci equations.*

3.2 The Codazzi-Mainardi equations

From (3.2) and (3.3) we immediately obtain the

Theorem 3.1. *(Codazzi-Mainardi integrability conditions)*
Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then the integrability conditions

$$(\partial_{uv}X_{u^i} - \partial_{vu}X_{u^i})^{\mathrm{norm}} \equiv 0$$

are the Codazzi-Mainardi equations

$$\begin{aligned}\partial_v L_{\sigma,i1} + \Gamma_{i1}^1 L_{\sigma,12} + \Gamma_{i1}^2 L_{\sigma,22} + \sum_{\omega=1}^{n} L_{\omega,i1} T_{\omega,2}^{\sigma} \\ = \partial_u L_{\sigma,i2} + \Gamma_{i2}^1 L_{\sigma,11} + \Gamma_{i2}^2 L_{\sigma,12} + \sum_{\omega=1}^{n} L_{\omega,i2} T_{\omega,1}^{\sigma}\end{aligned} \tag{3.6}$$

for $i = 1,2$ *and* $\sigma = 1,2,\ldots,n$.

The special case $n = 1$ of one codimension follows immediately with $T_{\sigma,i}^{\vartheta} \equiv 0$:

$$\partial_v L_{i1} + \Gamma_{i1}^1 L_{12} + \Gamma_{i1}^2 L_{22} = \partial_u L_{i2} + \Gamma_{i2}^1 L_{11} + \Gamma_{i2}^2 L_{12} \quad \text{for } i = 1,2.$$

These equations are named after the Italian mathematicians Gaspare Mainardi (*1800 in Milano; †1879 in Lecco) and Delfino Codazzi (*1824 in Lodi; †1873 in Pavia) who found them independently.

3.3 Gauss integrability conditions

Next, the equations (3.2) and (3.3) imply also the

Theorem 3.2. *(Gauss integrability conditions)*
Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ *together with an ONF* N *be given. Then the integrability conditions*

$$(\partial_{uv} X_{u^i} - \partial_{vu} X_{u^i})^{\text{tang}} \equiv 0$$

are the Gauss integrability equations

$$\Gamma_{i1,v}^{\ell} - \Gamma_{i2,u}^{\ell} + \sum_{m=1}^{2} (\Gamma_{i1}^m \Gamma_{m2}^{\ell} - \Gamma_{i2}^m \Gamma_{m1}^{\ell}) = \sum_{\ell,m=1}^{2} \sum_{\sigma=1}^{n} (L_{\sigma,i1} L_{\sigma,2m} - L_{\sigma,i2} L_{\sigma,1m}) g^{m\ell} \tag{3.7}$$

for $i,\ell = 1,2$.

Note that these conditions *do not involve the torsion coefficients.* We say they are related to the *inner geometry of a surface* while the torsions are quantities arising from the *embedding* of an immersion in space, i.e. they depend on the behaviour of its normal components.

Thus the Gauss integrability conditions in case $n = 1$ of one codimension read simply

$$\Gamma_{i1,v}^{\ell} - \Gamma_{i2,u}^{\ell} + \sum_{m=1}^{2} (\Gamma_{i1}^m \Gamma_{m2}^{\ell} - \Gamma_{i2}^m \Gamma_{m1}^{\ell}) = \sum_{\ell,m=1}^{2} (L_{i1} L_{2m} - L_{i2} L_{1m}) g^{m\ell}\,.$$

A good reference is Blaschke and Leichtweiss [15].

3.4 The curvature tensor

Now the left hand side of (3.7) gives rise to the following definition.

Definition 3.1. The *curvature tensor* R^{ℓ}_{ijk} of the immersion X is defined by

$$R^{\ell}_{ijk} := \partial_{u^k}\Gamma^{\ell}_{ij} - \partial_{u^j}\Gamma^{\ell}_{ik} + \sum_{m=1}^{2}(\Gamma^{m}_{ij}\Gamma^{\ell}_{mk} - \Gamma^{m}_{ik}\Gamma^{\ell}_{mj}) \tag{3.8}$$

for $i,j,k,\ell = 1,2$. It is also called *Riemann tensor* or *Riemann-Christoffel tensor.*

Then (3.7) can be written in the form

$$R^{\ell}_{i12} = \sum_{\ell,m=1}\sum_{\sigma=1}^{n}(L_{\sigma,i1}L_{\sigma,2m} - L_{\sigma,i2}L_{\sigma,1m})g^{m\ell}\,.$$

The *covariant components* of R^{ℓ}_{ijk} are defined as

$$R_{nijk} = \sum_{\ell=1}^{2} R^{\ell}_{ijk}g_{\ell n}\,. \tag{3.9}$$

Note finally that (for twodimensional surfaces) the components R_{nijk} *reduce essentially to the one single quantity* R_{2112}. We particularly compute

$$\begin{aligned}
&R_{1111} = 0, \quad R_{2222} = 0, \quad R_{1222} = 0, \quad R_{2111} = 0, \quad R_{2221} = 0, \quad R_{1112} = 0,\\
&R_{1122} = 0, \quad R_{2211} = 0, \quad R_{1121} = 0, \quad R_{1211} = 0, \quad R_{2212} = 0, \quad R_{2122} = 0,\\
&R_{2112} = R_{1221} = -R_{2121} = -R_{1212}\,.
\end{aligned}$$

As we will see shortly, this component R_{2112} represents exactly the Gauss curvature K (modulo the area element). This is the contents of the theorema egregium.

Regarding higher dimensional manifolds we would eventually face a fully occupied curvature tensor, and commonly one defines various traces of R^{ℓ}_{ijk}, for example the *Ricci curvature tensor*

$$R_{ij} := \sum_{m=1}^{2} R^{m}_{imj}$$

and the *Ricci scalar curvature*

$$R := \sum_{m,n=1}^{2} g^{mn}R_{mn}\,.$$

The German mathematician Bernhard Riemann (*1826 in Breselenz; †1866 in Verbania) was the first who introduced a conclusive concept of curvatures for *geometric manifolds of arbitrary dimension without referring to their embedding in any surrounding space.*

3.5 theorema egregium

Consider the Riemannian curvature tensor with the special indices $n=2$, $i=1$, $j=1$, $k=2$. Then from (3.7) and (3.9) we deduce the following fundamental result.

Theorem 3.3. *(theorema egregium)*
Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ be given. Then it holds

$$R_{2112}=KW^2 \tag{3.10}$$

with the Gaussian curvature K of the immersion X and its area element W.

Proof. We compute

$$\begin{aligned}R_{2112}&=\sum_{\ell=1}^{2}R_{112}^{\ell}g_{\ell 2}=\sum_{\ell,m=1}^{2}\sum_{\sigma=1}^{n}(L_{\sigma,11}L_{\sigma,2m}-L_{\sigma,12}L_{\sigma,1m})g^{m\ell}g_{\ell 2}\\&=\sum_{\ell=1}^{2}\sum_{\sigma=1}^{n}(L_{\sigma,11}L_{\sigma,22}-L_{\sigma,12}L_{\sigma,12})g^{2\ell}g_{\ell 2}=\sum_{\sigma=1}^{n}K_\sigma W^2=KW^2\,,\end{aligned}$$

and the statement follows. □

This *theorema egregium* (“wonderful theorem”) states that the Gauss curvature K can be expressed in terms of the coefficients g_{ij} of the surface’s first fundamental form and its first and second derivatives, encoded by the Christoffel symbols.

Thus K does not depend on the embedding of X in space.

Compare this result with the definition of the Gauss curvature from section 2.4 using normal frames which actually represent the way of the surface’s embedding in the surrounding Euclidean space!

In later applications we will use the following conformal representation of K.

Corollary 3.1. *Using conformal parameters $(u,v)\in B$, the Gaussian curvature K takes the form*

$$K=-\frac{1}{W}\,\Delta\log\sqrt{W} \tag{3.11}$$

with the area element W and the Euclidean Laplacian Δ.

Proof. Using (3.8) and (3.10) we compute

$$\begin{aligned}KW^2=R_{2112}&=\sum_{\ell=1}^{2}R_{112}^{\ell}g_{\ell 2}=R_{112}^{2}W\\&=\left\{\Gamma_{11,v}^{2}-\Gamma_{12,u}^{2}+\sum_{m=1}^{2}\Gamma_{11}^{m}\Gamma_{m2}^{2}-\sum_{m=1}^{2}\Gamma_{12}^{m}\Gamma_{m1}^{2}\right\}W.\end{aligned}$$

Thus together with the representations from (2.6) we arrive at

$$KW^2 = -\frac{W}{2}\left\{\left(\frac{W_u}{W}\right)_u + \left(\frac{W_v}{W}\right)_v\right\}$$

proving the statement. □

3.6 The Ricci equations

In case of one codimension, the normal components in (3.4) and (3.5), i.e.

$$\sum_{j,k=1}^{2} L_{1j}L_{k2}g^{jk} = \sum_{j,k=1}^{2} L_{2j}L_{k1}g^{jk},$$

are trivially satisfied taking the symmetry of the coefficients g^{ij} into account. But in case of higher codimension we obtain the following new information.

Theorem 3.4. *(Ricci integrability conditions)*
Let the immersion $X\colon B \longrightarrow \mathbb{R}^{n+2}$ together with an ONF N be given. Then the integrability conditions

$$(\partial_{uv}N_\sigma - \partial_{vu}N_\sigma)^{\text{norm}} \equiv 0$$

are the Ricci equations

$$T_{\sigma,2,u}^{\omega} - T_{\sigma,1,v}^{\omega} + \sum_{\vartheta=1}^{n} T_{\sigma,2}^{\vartheta}T_{\vartheta,1}^{\omega} - \sum_{\vartheta=1}^{n} T_{\sigma,1}^{\vartheta}T_{\vartheta,2}^{\omega} = \sum_{j,k=1}^{2}(L_{\sigma,2j}L_{\omega,k1} - L_{\sigma,1j}L_{\omega,k2})g^{jk} \tag{3.12}$$

for $\sigma, \omega = 1, \ldots, n$.

We immediately verify that the right hand side of this identity vanishes identically in case $n = 1$, i.e.

$$\sum_{j,k=1}^{2}(L_{2j}L_{k1} - L_{1j}L_{k2})g^{jk} = \sum_{j,k=1}^{2}(L_{2j}L_{j1} - L_{1j}L_{k2})g^{jk} = 0.$$

3.7 The curvature tensor of the normal bundle

The left hand side in (3.12) invites us to define a curvature of the normal bundle analogously to our definition of the Riemannian curvature tensor in terms of the Christoffel symbols

$$R_{ijk}^{\ell} = \partial_{u^k}\Gamma_{ij}^{\ell} - \partial_{u^j}\Gamma_{ik}^{\ell} + \sum_{m=1}^{2}(\Gamma_{ij}^{m}\Gamma_{mk}^{\ell} - \Gamma_{ik}^{m}\Gamma_{mj}^{\ell}).$$

The normal space of a surface at $w \in B$ was introduced as

$$\mathfrak{N}_X(w) = \left\{ Z \in \mathbb{R}^{n+2} \, : \, Z \cdot X_u(w) = Z \cdot X_v(w) = 0 \right\}.$$

Definition 3.2. The *curvature tensor* $S^{\vartheta}_{\sigma,ij}$ *of the normal bundle*

$$\mathfrak{N}_X := \bigcup_{w \in B} \mathfrak{N}_X(w)$$

of the immersion $X \colon B \to \mathbb{R}^{n+2}$ is defined as (notice the change of the signs)

$$\begin{aligned} S^{\omega}_{\sigma,ij} &:= \partial_{u^j} T^{\omega}_{\sigma,i} - \partial_{u^i} T^{\omega}_{\sigma,j} + \sum_{\vartheta=1}^{n} (T^{\vartheta}_{\sigma,i} T^{\omega}_{\vartheta,j} - T^{\vartheta}_{\sigma,j} T^{\omega}_{\vartheta,i}) \\ &= \sum_{m,n=1}^{2} (L_{\sigma,im} L_{\omega,jn} - L_{\sigma,jm} L_{\omega,in}) g^{mn} \end{aligned} \tag{3.13}$$

for $i, j = 1, 2$ and $\sigma, \omega = 1, \ldots, n$.

The second identity in (3.13) is due to the Ricci equations.

We want to draw the reader's attention to the skew-symmetry of the coefficients $S^{\omega}_{\sigma,ij}$ w.r.t. interchanging i and j, i.e.

$$S^{\omega}_{\sigma,12} = -S^{\omega}_{\sigma,21}$$

which enables us to concentrate on the components $S^{\omega}_{\sigma,12}$, and secondly to the skew-symmetry w.r.t. interchanging the indices σ and ω

$$S^{\omega}_{\sigma,12} = -S^{\sigma}_{\omega,12}.$$

It follows immediately that *in case* $n = 2$ *of two codimensions there is just one essential component, say* $S^{2}_{1,12}$. We faced a similar situation when we introduced the Gaussian curvature as the only essential component of the Riemannian curvature tensor. A detailled discussion of this fact follows in later chapters.

For the moment we want to present some elementary properties of the curvature tensor of the normal bundle. We already know the

Corollary 3.2. *The Ricci integrability conditions (3.12) from the last section can be rewritten in the form*

$$S^{\omega}_{\sigma,12} = \sum_{m,n=1}^{2} (L_{\sigma,1m} L_{\omega,n2} - L_{\sigma,2m} L_{\omega,n1}) g^{mn} \tag{3.14}$$

for $\sigma, \omega = 1, \ldots, n$.

We want to prove the tensorial transformation behaviour of the components $S_{\sigma,12}^{\vartheta}$.

Proposition 3.1. *Let $u^i(v^\alpha) \in \mathfrak{P}$. Then there hold*

$$S_{\sigma,k\ell}^{\omega} = \sum_{\kappa,\lambda=1}^{2} S_{\sigma,\kappa\lambda}^{\omega} \Lambda_k^\kappa \Lambda_\ell^\lambda$$

for all $k,\ell = 1,2$ and $\sigma,\omega = 1,\ldots,n$.

Proof. We calculate

$$\begin{aligned} S_{\sigma,k\ell}^{\omega} &= \sum_{m,n=1}^{2} (L_{\sigma,km}L_{\omega,\ell n} - L_{\sigma,\ell m}L_{\omega,kn})g^{mn} \\ &= \sum_{m,n=1}^{2} \sum_{\kappa,\lambda,\mu,\nu=1}^{2} (L_{\sigma,\kappa\mu}L_{\omega,\lambda\nu} - L_{\sigma,\lambda\mu}L_{\omega,\kappa\nu})g^{\mu'\nu'} \Lambda_k^\kappa \Lambda_\ell^\lambda \Lambda_m^\mu \Lambda_n^\nu \overline{\Lambda}_{\mu'}^m u \overline{\Lambda}_{\nu'}^n \\ &= \sum_{\kappa,\lambda,\mu,\nu=1}^{2} (L_{\sigma,\kappa\mu}L_{\omega,\lambda\nu} - L_{\sigma,\lambda\mu}L_{\omega,\kappa\nu})g^{\mu\nu} \Lambda_k^\kappa \Lambda_\ell^\lambda \,, \end{aligned}$$

which proves the statement. □

Although *the components $S_{\sigma,12}^{\omega}$ are not invariant w.r.t. $SO(n)$-actions* they can be considered as *sectional curvatures* in the following sense.

Proposition 3.2. *Let $\sigma \neq \omega$. The curvature components $S_{\sigma,12}^{\omega}$ do not depend on the choice of an orthonormal basis of the sectional plane* Span$\{N_\sigma, N_\omega\}$.

We particularly infer *if $S_{\sigma,12}^{\omega} \equiv 0$ holds for one chosen ONF N then it is also true for all ONF.* This leads us to a central notion of our investigations.

Definition 3.3. The normal bundle $\mathfrak{N}_X$ is called *flat* if and only there hold

$$S_{\sigma,12}^{\omega} \equiv 0 \quad \text{for all } \sigma,\omega = 1,\ldots,n$$

w.r.t. to some ONF N.

Now we come to the proof of the foregoing proposition.

Proof. We introduce conformal parameter $(u,v) \in B$. Let again

$$\widetilde{N}_\sigma = \cos\varphi\, N_\sigma + \sin\varphi\, N_\omega \,, \quad \widetilde{N}_\omega = -\sin\varphi\, N_\sigma + \cos\varphi\, N_\omega \,,$$

and insert it into the representation of $S_{\sigma,12}^{\omega}$ using Ricci's integrability conditions.

Then we compute

$$\begin{aligned}
W\widetilde{S}^{\omega}_{\sigma,12} &= (\widetilde{L}_{\sigma,11}\widetilde{L}_{\omega,12} - \widetilde{L}_{\sigma,21}\widetilde{L}_{\omega,11}) + (\widetilde{L}_{\sigma,12}\widetilde{L}_{\omega,22} - \widetilde{L}_{\sigma,22}\widetilde{L}_{\omega,21}) \\
&= (\cos\varphi\, L_{\sigma,11} + \sin\varphi\, L_{\omega,11})(-\sin\varphi\, L_{\sigma,12} + \cos\varphi\, L_{\omega,12}) \\
&\quad - (\cos\varphi\, L_{\sigma,21} + \sin\varphi\, L_{\omega,21})(-\sin\varphi\, L_{\sigma,11} + \cos\varphi\, L_{\omega,11}) \\
&\quad + (\cos\varphi\, L_{\sigma,12} + \sin\varphi\, L_{\omega,12})(-\sin\varphi\, L_{\sigma,22} + \cos\varphi\, L_{\omega,22}) \\
&\quad - (\cos\varphi\, L_{\sigma,22} + \sin\varphi\, L_{\omega,22})(-\sin\varphi\, L_{\sigma,21} + \cos\varphi\, L_{\omega,21}) \\
&= (L_{\sigma,11} - L_{\sigma,22})L_{\omega,12} - (L_{\omega,11} - L_{\omega,22})L_{\sigma,12} = WS^{\omega}_{\sigma,12}\,,
\end{aligned}$$

which proves the statement. □

The next result reveals an interesting connection between the curvature components $S^{\vartheta}_{\sigma,12}$ and normal mean curvature quantities $H_{\sigma\vartheta}$ introduced in section 2.5 of the previous chapter.

Theorem 3.5. *There hold*

$$S^{\vartheta}_{\sigma,12} \equiv \frac{1}{2} H_{\sigma\vartheta} W \tag{3.15}$$

for $\sigma, \vartheta = 1, \ldots, n$.

Proof. Recall the definition of the coefficients $L_{\sigma\vartheta,ij}$ from Definition 2.6. Using conformal parameters we calculate

$$\begin{aligned}
H_{\sigma\vartheta} &= \frac{1}{W^2}(L_{\sigma\vartheta,11} + L_{\sigma\vartheta,22}) \\
&= \frac{2}{W^2}(L_{\sigma,11}L_{\vartheta,12} - L_{\sigma,12}L_{\vartheta,11} + L_{\sigma,12}L_{\vartheta,22} - L_{\sigma,22}L_{\vartheta,12}) = \frac{2}{W} S^{\vartheta}_{\sigma,12}
\end{aligned}$$

proving the identity. □

3.8 The curvature of the normal bundle

Let $u^i = u^i(v^\alpha) \in \mathfrak{P}$. Then Proposition 3.1 yields

$$S^{\vartheta}_{\sigma,12}(u^1,u^2) = S^{\vartheta}_{\sigma,12}(v^1,v^2)(\Lambda^1_1\Lambda^2_2 - \Lambda^2_1\Lambda^1_2).$$

Thus taking account of transformation behaviour of the area element $W(u^1,u^2)$ from Corollary 1.1 we conclude *that the quantities*

$$\frac{1}{W(u^1,u^2)} S^{\vartheta}_{\sigma,12}(u^1,u^2)$$

are independent of the chosen parametrization.

This leads us to our next

Definition 3.4. Let the immersion $X\colon B\to\mathbb{R}^4$ be given. Then *the scalar curvature of its normal bundle* is defined as

$$S:=\frac{1}{W}\,S_{1,12}^2\,.$$

Note, as mentioned above, that $S_{1,12}^2$ is the only essential component of the curvature tensor if $n=2$. In case $n>2$ of higher codimensions we would like to work with a *curvature vector instead of a single curvature scalar.*

Definition 3.5. Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ be given. Then *the curvature vector of its normal bundle* is

$$S=\frac{1}{W}\,(S_{1,12}^2,S_{1,12}^3,\ldots,S_{1,12}^n,S_{2,12}^3,\ldots,S_{n-1,12}^n)\in\mathbb{R}^N$$

with $N=\binom{n}{2}$.

Let us conclude this section with two remarks.

1. Although all informations of the curvature tensor of the normal bundle are already decoded in the normal mean curvature matrix $\mathbf{H}$, we find it more natural to work with the vector $S\in\mathbb{R}^N$.
2. Worthwile for the future but omitted here is an elaborative investigation of *immersions with prescribed curvature vector of the normal bundle.*

3.9 Examples

We want to consider some interesting examples of surfaces with flat and with non-vanishing normal bundles.

Spherical surfaces in S^3

Suppose $|X(u,v)|\equiv 1$ for all $(u,v)\in B$. We immediately compute

$$X_u\cdot X=0,\quad X_v\cdot X=0,$$

i.e. the surface vector X itself serves as a unit normal vector, say $N_1:=X$. Choose then N_2 to complete the system $\{X_u,X_v,N_1\}$ to a basis of the embedding space $\mathbb{R}^4$. We immediately verify that the surface has flat normal bundle since

$$T_{1,1}^2=N_{1,u}\cdot N_2=X_u\cdot N_2=0,\quad T_{1,2}^2=N_{1,v}\cdot N_2=X_v\cdot N_2=0.$$

The Clifford torus

This is the surface
$$X=\frac{1}{\sqrt{2}}\,(\cos u,\sin u,\cos v,\sin v)$$
resembling $S^1\times S^1$. We assign the moving 4-frame $\{X_u,X_v,N_1,N_2\}$ consisting of the tangential vectors
$$X_u=\frac{1}{\sqrt{2}}\,(-\sin u,\cos u,0,0),\quad X_v=\frac{1}{\sqrt{2}}\,(0,0,-\sin v,\cos v)$$
as well as two unit normal vectors
$$N_1:=\frac{1}{\sqrt{2}}\,(\cos u,\sin u,\cos v,\sin v),\quad N_2:=\frac{1}{\sqrt{2}}\,(-\cos u,-\sin u,\cos v,\sin v).$$

Obviously the torsion coefficients of this ONF vanish identically, thus the normal bundle of this immersion is flat; see e.g. do Carmo [22] for this example.

For explicit constructions of surfaces with flat normal bundles we want to refer the reader to Ferapontov [57].

Parallel type surfaces

Consider now the normal transport
$$R(u,v)=X(u,v)+f(u,v)N_1(u,v)+g(u,v)N_2(u,v).$$
If the functions f and g are constant then we say *R is the parallel surface of X* and vice versa, at least if the surfaces are immersed in $\mathbb{R}^3$. Parallelity in higher codimensional space depends on the curvature S of the normal bundle.

Proposition 3.3. *The normal transport R of an immersion $X:\overline{B}\to\mathbb{R}^4$ is parallel, i.e. there holds*
$$R_{u^i}\cdot N_\sigma=0\quad \textit{for } i=1,2,\ \sigma=1,2,$$
if and only if it holds $S\equiv 0$ for the scalar curvature S of the normal bundle.

Proof. For the proof we use the Weingarten equations and compute the normal parts $R_u^\perp$ and $R_v^\perp$ of the tangential vectors R_u resp. R_v,
$$R_u^\perp=f_uN_1+g_uN_2+fN_{1,u}^\perp+gN_{2,u}^\perp\,=\,(f_u-gT_{1,1}^2)N_1+\big(g_u+fT_{1,1}^2\big)N_2\,,$$
$$R_v^\perp=f_vN_1+g_vN_2+fN_{1,v}^\perp+gN_{2,v}^\perp\,=\,\big(f_v-gT_{1,2}^2\big)N_1+\big(g_v+fT_{1,2}^2\big)N_2\,.$$
The condition of parallelity leads us to the first order system
$$f_u-gT_{1,1}^2=0,\quad f_v-gT_{1,2}^2=0,\quad g_u+fT_{1,1}^2=0,\quad g_v+fT_{1,2}^2=0.$$

Differenting the first two equations and making use of the other two conditions shows us

$$0 = f_{uv} - g_v T^2_{1,1} - g T^2_{1,1,v} = f_{uv} + f T^2_{1,1} T^2_{1,2} - g T^2_{1,1,v},$$

$$0 = f_{vu} - g_u T^2_{1,2} - g T^2_{1,2,u} = f_{vu} + f T^2_{1,1} T^2_{1,2} - g T^2_{1,2,u},$$

thus a comparison of the right hand sides brings

$$0 = -g T^2_{1,1,v} + g T^2_{1,2,u} = -g \cdot SW.$$

Similarly we find $0 = f \cdot SW$, which proves the statement. □

Parallel type surface are widely used in geometry and mathematical physics. We want to refer the reader to da Costa [37] for an application in quantum mechanics in curved spaces.

3.10 The fundamental theorem

We want to reconstruct an immersion X from given first and second fundamental forms, given torsion coefficients, and a given $(n+2)$-frame attached at some point of the surface, say at $(0,0) \in B$. The latter assumption is needed to construct an initial $(n+2)$-frame at an arbitrary point of the surface.

In particular, it is needed in the second point of our proof below: Assume we know that the tangential planes at a certain point $w \in B$ of two solutions X and $\widetilde{X}$ coincide, and therefore the normal spaces are the same. If $n = 1$, we are then allowed to infer that the unit normal vectors also coincide (up to orientation).

But the situation is more involved in general: Even if the normal spaces of X and $\widetilde{X}$ agree at some point, we have no further information about the behaviour of the respective normal frames. We will fix this problem under this special assumption.

Theorem 3.6. *(Fundamental theorem of surface theory)*

Assumptions: Let us given

1. *a quadratic, symmetric, and positive definite form* $\widehat{g}_{ij} \in C^3(B,\mathbb{R})$, *such that*

$$\widehat{g}_{ij} = \widehat{g}_{ji}, \quad \det(\widehat{g}_{ij})_{i,j=1,2} > 0; \tag{3.16}$$

2. *n symmetric and quadratic forms* $\widehat{L}_{\sigma,ij} \in C^2(B,\mathbb{R})$, *such that*

$$\widehat{L}_{\sigma,ij} = \widehat{L}_{\sigma,ji} \tag{3.17}$$

for $i,j = 1,2$ *and* $\sigma = 1,\ldots,n$;

3. *and* $\binom{n}{2}$ *forms* $\widehat{T}^{\vartheta}_{\sigma,i} \in C^2(B,\mathbb{R})$ *such that*

$$\widehat{T}^{\vartheta}_{\sigma,i} = -\widehat{T}^{\vartheta}_{\sigma,i} \tag{3.18}$$

for $i = 1,2$ *and* $\sigma, \vartheta = 1,\ldots,n$.

Assume that these forms satisfy the integrability conditions (3.6), (3.7), and (3.12).

4. *Finally, let* $n+2$ *vectors* $\mathring{Z}_{(1)}$, $\mathring{Z}_{(2)}$, *and* $\mathring{N}_1,\ldots,\mathring{N}_n$ *be given with the properties*

$$\mathring{Z}_{(i)} \cdot \mathring{Z}_{(j)} = \widehat{g}_{ij}(0,0), \quad \mathring{Z}_{(i)} \cdot \mathring{N}_\sigma = 0, \quad \mathring{N}_\sigma \cdot \mathring{N}_\vartheta = \delta_{\sigma\vartheta} \tag{3.19}$$

for $i = 1,2$ *and* $\sigma, \vartheta = 1,\ldots,n$.

Statement: Then there is a unique immersion $X \in C^3(B,\mathbb{R}^{n+2})$ *with an unique ONF* $N = (N_1,\ldots,N_n)$ *such that*

$$X(0,0) = \mathring{X}, \quad X_{u^i}(0,0) = \mathring{Z}_{(i)}, \quad N_\sigma(0,0) = \mathring{N}_\sigma .$$

The $g_{ij} \equiv \widehat{g}_{ij}$ *agrees with the first fundamental form of the surface* X, *the* $L_{\sigma,ij} \equiv \widehat{L}_{\sigma,ij}$ *are the coefficients of its second fundamental forms w.r.t. to the ONF* N, *and the* $T^{\vartheta}_{\sigma,i} \equiv \widehat{T}^{\vartheta}_{\sigma,i}$ *represent the respective torsion coefficients.*

Our proof of this theorem follows the lines of Blaschke and Leichtweiss [15]. Preceded is the following lemma (see also ibidem, § 60).

Lemma 3.1. *Consider the initial value problem*

$$\frac{\partial z_k}{\partial u^i} = \sum_{\ell=1}^{m} a^{\ell}_{ki} z_\ell\,, \quad z_k(u_0,v_0) = \mathring{z}_k\,, \quad i = 1,2,\ k = 1,\ldots,m, \tag{3.20}$$

with $m \ge 1$ *linear partial differential equations for the unknowns* z_ℓ *on a right-angled domain* $G : |u^i - u^i_0| < b_i$, $i = 1,2$. *Assume that* $a^{\ell}_{ki} \in C^2(G,\mathbb{R})$ *for all* $i = 1,2$ *and* $k,\ell = 1,\ldots,m$. *Then the following hold true:*

1. *There is at most one solution vector* $(z_1,\ldots,z_m) \in C^3(G,\mathbb{R}^m)$ *of the initial value problem (3.20).*
2. *The system of the linear equations is solvable if and only if the integrability conditions*
$$\frac{\partial^2 z_k}{\partial u^i \partial u^j} - \frac{\partial^2 z_k}{\partial u^j \partial u^i} \equiv 0$$
are satisfied for all $k = 1,\ldots,m$.

We particularly compute

$$0 \equiv \frac{\partial}{\partial u^j} \sum_{\ell=1}^{m} a_{ki}^{j} z_\ell - \frac{\partial}{\partial u^i} \sum_{\ell=1}^{m} a_{kj}^{\ell} z_\ell = \sum_{\ell=1}^{m} \left\{ \frac{\partial a_{ki}^{\ell}}{\partial u^j} - \frac{\partial a_{kj}^{\ell}}{\partial u^i} + \sum_{s=1}^{m} a_{ki}^{s} a_{sj}^{\ell} - \sum_{s=1}^{m} a_{kj}^{s} a_{si}^{\ell} \right\} z_\ell \, .$$

Thus the system is solvable if there hold the integrability conditions

$$\frac{\partial a_{ki}^{\ell}}{\partial u^j} + \sum_{s=1}^{m} a_{ki}^{s} a_{sj}^{\ell} = \frac{\partial a_{kj}^{\ell}}{\partial u^i} + \sum_{s=1}^{m} a_{kj}^{s} a_{si}^{\ell} \quad \text{for all } k, \ell = 1, \ldots, m.$$

Proof of the theorem. 1. Rewriting the differential equations (2.2) and (2.11) of Gauss and Weingarten we arrive at the linear system

$$\begin{aligned} X_{u^i} &= Z_{(i)} \, , \\ Z_{(i),u^j} &= \sum_{k=1}^{2} \Gamma_{ij}^{k} Z_{(k)} + \sum_{\vartheta=1}^{n} L_{\vartheta,ij} N_\vartheta \, , \\ N_{\sigma,u^i} &= -\sum_{j,k=1}^{2} L_{\sigma,ij} g^{jk} Z_{(k)} + \sum_{\vartheta=1}^{n} T_{\sigma,i}^{\vartheta} N_\vartheta \end{aligned} \tag{3.21}$$

for the unknown vector functions X, $Z_{(1)}$, $Z_{(2)}$, N_1, $N_2, \ldots, N_n$.

2. Assume that there are two immersed solutions X and $\widetilde{X}$. At $(0,0) \in B$ there hold

$$\widetilde{X}_{u^i}(0,0) \cdot \widetilde{X}_{u^j}(0,0) = g_{ij}(0,0) = X_{u^i}(0,0) \cdot X_{u^j}(0,0) \quad \text{for } i, j = 1, 2.$$

After a suitable translation and rotation of $\widetilde{X}$ we can arrange the geometry so that

$$\widetilde{X}(0,0) = X(0,0), \quad \widetilde{X}_{u^i}(0,0) = X_{u^i}(0,0) \quad \text{for } i = 1, 2.$$

Together with (3.19) we conclude that in $(0,0)$ the tangential vectors, the ONF N and ONF $\widetilde{N}$ as well as their orientations agree. Lemma 3.1 proves the uniqueness.

3. To prove that a solution X of (3.21) is indeed immersed with the prescribed forms (3.16), (3.17), (3.18) and the given $(n+2)$-frame (3.19) in $(0,0) \in B$, we will show that the following functions vanish identically in B (by assumption they vanish at $(0,0) \in B$)

$$\begin{aligned} f_{ij}^{(1)} &:= Z_{(i)} \cdot Z_{(j)} - \widehat{g}_{ij} && \text{for } i, j = 1, 2, \\ f_{i\sigma}^{(2)} &:= Z_{(i)} \cdot N_\sigma && \text{for } i = 1, 2, \ \sigma = 1, \ldots, n, \\ f_{\sigma}^{(3)} &:= N_\sigma^2 - 1 && \text{for } \sigma = 1, \ldots, n, \\ f_{\sigma\vartheta}^{(4)} &:= N_\sigma \cdot N_\vartheta && \text{for } \sigma, \vartheta = 1, \ldots, n, \ \sigma \neq \vartheta. \end{aligned}$$

4. For this purpose we establish a linear system for all these functions. First, (3.21) yields

$$\begin{aligned}\left\{Z_{(i)}\cdot Z_{(j)}-\widehat{g}_{ij}\right\}_{u^k} &= Z_{(i),u^k}\cdot Z_{(j)}+Z_{(j),u^k}\cdot Z_{(i)}-\widehat{g}_{ij,u^k}\\ &= \sum_{m=1}^{2}\widehat{\Gamma}_{ik}^{m}Z_{(m)}\cdot Z_{(j)}+\sum_{m=1}^{2}\widehat{\Gamma}_{jk}^{m}Z_{(m)}\cdot Z_{(i)}-\widehat{g}_{ij,u^k}\\ &\quad+\sum_{\vartheta=1}^{n}\widehat{L}_{\vartheta,ik}\left\{N_\vartheta\cdot Z_{(j)}\right\}+\sum_{\vartheta=1}^{n}\widehat{L}_{\vartheta,jk}\left\{N_\vartheta\cdot Z_{(i)}\right\}\\ &= \sum_{m=1}^{2}\widehat{\Gamma}_{ik}^{m}\left\{Z_{(m)}\cdot Z_{(j)}-\widehat{g}_{mj}\right\}+\sum_{m=1}^{2}\widehat{\Gamma}_{jk}^{m}\left\{Z_{(m)}\cdot Z_{(i)}-\widehat{g}_{mi}\right\}\\ &\quad+\sum_{\vartheta=1}^{n}\widehat{L}_{\vartheta,ik}\left\{Z_{(j)}\cdot N_\vartheta\right\}+\sum_{\vartheta=1}^{n}\widehat{L}_{\vartheta,jk}\left\{Z_{(i)}\cdot N_\vartheta\right\}\end{aligned} \tag{3.22}$$

taking

$$\begin{aligned}\sum_{m=1}^{2}\widehat{\Gamma}_{ik}^{m}\widehat{g}_{mj}+\sum_{m=1}^{2}\widehat{\Gamma}_{jk}^{m}\widehat{g}_{mi} &= \frac{1}{2}\sum_{m,n=1}^{2}\widehat{g}^{mn}(\widehat{g}_{ni,u^k}+\widehat{g}_{kn,u^i}-\widehat{g}_{ik,u^n})\widehat{g}_{mj}\\ &\quad+\frac{1}{2}\sum_{m,n=1}^{2}\widehat{g}^{mn}(\widehat{g}_{nj,u^k}+\widehat{g}_{kn,u^j}-\widehat{g}_{jk,u^n})\widehat{g}_{mi}\\ &= \frac{1}{2}(\widehat{g}_{ij,u^k}+\widehat{g}_{kj,u^i}-\widehat{g}_{ik,u^j})+\frac{1}{2}(\widehat{g}_{ij,u^k}+\widehat{g}_{ki,u^j}-\widehat{g}_{jk,u^i})\\ &= \widehat{g}_{ij,u^k}\end{aligned}$$

into account. The functions $Z_{(i)}\cdot N_\sigma$ satisfy

$$\begin{aligned}\left\{Z_{(i)}\cdot N_\sigma\right\}_{u^k} &= Z_{(i),u^k}\cdot N_\sigma+Z_{(i)}\cdot N_{\sigma,u^k}\\ &= \widehat{\Gamma}_{ik}^{m}\left\{Z_{(m)}\cdot N_\sigma\right\}+\widehat{L}_{\sigma,ik}N_\sigma^2-\widehat{L}_{\sigma,km}\widehat{g}^{mn}\left\{Z_{(i)}\cdot Z_{(n)}\right\}\\ &\quad+\sum_{\substack{\vartheta=1\\ \sigma\neq\vartheta}}^{n}\widehat{L}_{\vartheta,ik}\left\{N_\sigma\cdot N_\vartheta\right\}+\sum_{\vartheta=1}^{n}\widehat{T}_{\sigma,k}^{\vartheta}\left\{Z_{(i)}\cdot N_\vartheta\right\}\\ &= \widehat{\Gamma}_{ik}^{m}\left\{Z_{(m)}\cdot N_\sigma\right\}+\widehat{L}_{\sigma,ik}\left\{N_\sigma^2-1\right\}-\widehat{L}_{\sigma,km}\widehat{g}^{mn}\left\{Z_{(i)}\cdot Z_{(n)}-\widehat{g}_{in}\right\}\\ &\quad+\sum_{\substack{\vartheta=1\\ \sigma\neq\vartheta}}^{n}\widehat{L}_{\vartheta,ik}\left\{N_\sigma\cdot N_\vartheta\right\}+\sum_{\vartheta=1}^{n}\widehat{T}_{\sigma,k}^{\vartheta}\left\{Z_{(i)}\cdot N_\vartheta\right\}\end{aligned} \tag{3.23}$$

due to the property

$$\sum_{m,n=1}^{2} \widehat{L}_{\sigma,km}\widehat{g}^{mn}\widehat{g}_{ni} = \sum_{m=1}^{2} \widehat{L}_{\sigma,km}\delta_i^m = \widehat{L}_{\sigma,ki}\,.$$

Next we have

$$\begin{aligned}\left\{N_\sigma^2 - 1\right\}_{u^k} &= 2\,N_\sigma \cdot N_{\sigma,u^k} \\ &= -2\sum_{m,n=1}^{2} \widehat{L}_{\sigma,km}\widehat{g}^{mn}\left\{Z_{(n)}\cdot N_\sigma\right\} + 2\sum_{\vartheta=1}^{n} \widehat{T}_{\sigma,k}^{\vartheta}\left\{N_\sigma\cdot N_\vartheta\right\}.\end{aligned} \tag{3.24}$$

Finally we calculate

$$\begin{aligned}\left\{N_\sigma\cdot N_\vartheta\right\}_{u^k} &= N_{\sigma,u^k}\cdot N_\vartheta + N_\sigma\cdot N_{\vartheta,u^k} \\ = -&\sum_{m,n=1}^{2} \widehat{L}_{\sigma,km}\widehat{g}^{mn}\left\{Z_{(n)}\cdot N_\vartheta\right\} - \sum_{m,n=1}^{2} \widehat{L}_{\vartheta,km}\widehat{g}^{mn}\left\{Z_{(n)}\cdot N_\sigma\right\} \\ &+ \sum_{\omega=1}^{n} \widehat{T}_{\sigma,k}^{\omega}\left\{N_\omega\cdot N_\vartheta\right\} + \sum_{\omega=1}^{n} \widehat{T}_{\vartheta,k}^{\omega}\left\{N_\omega\cdot N_\sigma\right\}.\end{aligned} \tag{3.25}$$

6. Summarizing (3.22), (3.23), (3.24), (3.25) we arrive at the linear system

$$\begin{aligned} f_{ij,u^k}^{(1)} &= \sum_{m=1}^{2} \widehat{\Gamma}_{ik}^{m} f_{mj}^{(1)} + \sum_{m=1}^{2} \widehat{\Gamma}_{jk}^{m} f_{mi}^{(1)} + \sum_{\vartheta=1}^{n} \widehat{L}_{\vartheta,ik} f_{j\vartheta}^{(2)} + \sum_{\vartheta=1}^{n} \widehat{L}_{\vartheta,jk} f_{i\vartheta}^{(2)}\,, \\ f_{i\sigma,u^k}^{(2)} &= \sum_{m=1}^{2} \widehat{\Gamma}_{ik}^{m} f_{m\sigma}^{(2)} + \widehat{L}_{\sigma,ik} f_{\sigma}^{(3)} - \sum_{m,n=1}^{2} \widehat{L}_{\sigma,km}\widehat{g}^{mn} f_{in}^{(1)} + \sum_{\substack{\vartheta=1\\ \sigma\neq\vartheta}}^{n} \widehat{L}_{\vartheta,ik} f_{\sigma\vartheta}^{(4)} \\ &\quad + \sum_{\vartheta=1}^{n} \widehat{T}_{\sigma,k}^{\vartheta} f_{i\vartheta}^{(2)}\,, \\ f_{\sigma,u^k}^{(3)} &= -2\sum_{m,n=1}^{2} \widehat{L}_{\sigma,km}\widehat{g}^{mn} f_{n\sigma}^{(2)} + \sum_{\substack{\vartheta=1\\ \sigma\neq\vartheta}}^{n} \widehat{T}_{\sigma,k}^{\vartheta} f_{\sigma\vartheta}^{(4)}\,, \\ f_{\sigma\vartheta,u^k}^{(4)} &= -\sum_{m,n=1}^{2} \widehat{L}_{\sigma,km}\widehat{g}^{mn} f_{n\vartheta}^{(2)} - \sum_{m,n=1}^{2} \widehat{L}_{\vartheta,km}\widehat{g}^{mn} f_{n\sigma}^{(2)} + \sum_{\substack{\omega=1\\ \omega\neq\vartheta}}^{n} \widehat{T}_{\sigma,k}^{\omega} f_{\omega\vartheta}^{(4)} \\ &\quad + \sum_{\substack{\omega=1\\ \omega\neq\vartheta}}^{n} \widehat{T}_{\vartheta,k}^{\omega} f_{\omega\sigma}^{(4)} \end{aligned}$$

with the same initial conditions as the trivial solution $0,\ldots,0$. Thus the uniqueness stated in Lemma 3.1 proves that all the functions $f_{ij}^{(1)}$, $f_{i\sigma}^{(2)}$, $f_{\sigma}^{(3)}$ and $f_{\sigma\vartheta}^{(4)}$ vanish identically in B.

7. We complete our proof of the fundamental theorem as follows:

- Due to (3.21), (3.22) and the property

$$\sum_{i=1}^{2} \alpha^i X_{u^i} = 0 \quad \text{implies} \quad \alpha^1 = \alpha^2 = 0$$

  for arbitrary functions α^1, α^2 since

$$\left(\sum_{i=1}^{2} \alpha^i X_{u^i} \right) \cdot \left(\sum_{j=1}^{2} \alpha^j X_{u^j} \right) = \sum_{i,j=1}^{2} \alpha^i \alpha^j g_{ij} = 0,$$

  the mapping $X \in C^3(B, \mathbb{R}^{n+2})$ represents a two-dimensional immersion in $\mathbb{R}^{n+2}$ with first fundamental form $g_{ij} \equiv \widehat{g}_{ij}$.
- Following (3.23), (3.24), (3.25), the moving frame N forms an orthonormal normal frame which is orthogonal to the span of X_u and X_v.
- Therefore, $X_{u^i u^j} \cdot N_\sigma = \widehat{L}_{\vartheta,ij}$ for all $i, j = 1, 2$ and all $\sigma = 1, \ldots, n$, and the $\widehat{L}_{\sigma,ij}$ are detected as the coefficients of the second fundamental forms w.r.t. N.
- Analogously we prove that the $\widehat{T}_{\sigma,i}^{\vartheta}$ agree with the torsion coefficients w.r.t. the ONF N.

The proof is complete. □

This result was already proved in Weyl [168]. We want to refer the reader also to Brauner [18].

Chapter 4
Weighted differential geometry

In this chapter we extend the previous investigations and consider weighted fundamental forms of Finslerian type, the natural differential geometric setup for the calculus of variations.

4.1 Introduction

Consider an immersion $X\colon B \to \mathbb{R}^3$ in three-dimensional Euclidean space with unit normal vector $N \in \mathbb{R}^3$. We suppose that its first fundamental form, now denoted by

$$(h_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times 2}$$

to distinguish it from our investigations so far, *results from the action of a symmetric and positive definite matrix*

$$\mathbf{W}(X,Z) = \big(w_{ij}(X,Z)\big)_{i,j=1,2,3}\colon \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}^{3\times 3}$$

in the following way:

$$h_{ij} := X_{u^i} \circ \mathbf{W}(X,N) \circ X_{u^j}\,, \quad i,j = 1,2.$$

In particular, if $\mathbf{W}(X,Z) \equiv \mathbb{E}^3$ with the three-dimensional unit matrix $\mathbb{E}^3$, then $h_{ij} = g_{ij}$ with the coefficients

$$g_{ij} = X_{u^i} \cdot X_{u^j}\,, \quad i,j = 1,2$$

of the classical first fundamental from the previous chapters.

So far as we know, *weighted fundamental forms* h_{ij} of this special kind where first introduced by Sauvigny [141] for analytical studies of critical points $X\colon B \to \mathbb{R}^3$ for two-dimensional parametric and anisotropic variational problems

$$\mathscr{F}[X] := \iint\limits_B F(X_u \times X_v)\,du\,dv \longrightarrow \text{min!}$$

Critical points of $\mathscr{F}[X]$ are called *F-minimal surfaces*. In many respects they behave like ordinary minimal surfaces if certain classical geometric quantities are replaced by their *weighted counterparts* as demonstrated in chapter 11.

As we expound later, such weighted geometric quantities, like the quadratic form $(h_{ij})_{i,j=1,2}$ and various curvatures derived from it, furnish the natural setup for anisotropic variational problems.

There is a strong coincidence with the so-called Finsler spaces from the calculus of variations, named after the Swiss mathematician Paul Finsler (*1894 in Heilbronn; †1970 in Zürich). Excellent introductions to this matter can be found in Finsler's thesis [59], or in the textbook of Funk [68].

In this chapter we want to develop an approach to *a weighted differential geometry* for surfaces immersed in $\mathbb{R}^3$ and $\mathbb{R}^{n+2}$. For illustration consider the weighted line element of the form

$$ds_W^2 := \sum_{i,j=1}^{3} w_{ij}(X,Z)\,dx^i dx^j\,.$$

If the w_{ij} depend only on the space point X then the inhomogeneous metric ds_W^2 is called of *Riemannian type.* More generally, if ds_W^2 is *inhomogeneous and anisotropic*, i.e. the $w_{ij}(X,Z)$ depend additionally on a direction vector Z attached at the point X, we say ds_W^2 is of *Finslerian type.*

Consider now the space $\mathbb{R}^3$ equipped with such an inhomogeneous and anisotropic metric. Inserting the surface's representation $X = X(u,v)$ and its unit normal vector $N = N(u,v)$ into the form ds_W^2 yields

$$\begin{aligned} ds_W^2 &= \sum_{i,j=1}^{2} w_{ij}(X,N)\,dx^i dx^j = \sum_{i,j=1}^{2}\sum_{k,\ell=1}^{2} w_{ij}(X,N)x^i_{u^k}x^j_{u^\ell}\,du^k du^\ell \\ &= \sum_{k,\ell=1}^{2} h_{k\ell}\,du^k du^\ell \end{aligned}$$

setting

$$\begin{aligned} h_{k\ell} &:= \sum_{i,j=1}^{2} X_{u^i}\circ \mathbf{W}(X,N)\circ X_{u^j}\,, \quad k,\ell = 1,2, \\ \text{and} \quad & \mathbf{W}(X,Z) = (w_{ij}(X,Z))_{i,j=1,2,3}\,. \end{aligned}$$

But these h_{ij} are exactly the coefficients of the weighted first fundamental form from the beginning of this introduction!

It remains to remark that in the special case $\mathbf{W}(X,Z) \equiv \mathbb{E}^3$ we obviously retrieve the classical metrical form $(g_{ij})_{i,j=1,2}$.

4.2 The weighted fundamental forms

Consider an immersion $X\colon B \to \mathbb{R}^{n+2}$ with ONF N. We introduce a symmetric and positive definite matrix

$$\begin{aligned} &\mathbf{W}(X,Z) = (w_{i,j})_{i,j=1,\ldots,n+2} \\ &\text{of class} \quad C^3(\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\setminus\{0\}\times\ldots\times\mathbb{R}^{n+2}\setminus\{0\},\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}) \end{aligned} \tag{4.1}$$

satisfying the properties:

Let $Z = (Z_1,\ldots,Z_n)$ and $\widetilde{Z} = (\widetilde{Z}_1,\ldots,\widetilde{Z})$ be some orthogonal frames in the normal space of X, not necessarily consisting of unit vectors, then suppose

(W1) $\mathbf{W}(X,Z)$ is invariant w.r.t. the choise of the normal frame, i.e. it holds

$$\mathbf{W}(X,Z) = \mathbf{W}(X,\widetilde{Z}) \quad \text{for all ONF } Z \text{ and } \widetilde{Z};$$

(W2) $\mathbf{W}(X,Z)$ acts non-trivially only on the tangent space, i.e. there hold

$$\begin{aligned} &\mathbf{W}(X,Z)\big|_{\mathfrak{T}_X(w)}\colon \mathfrak{T}_X(w) \longrightarrow \mathfrak{T}_X(w), \quad \text{in particular,} \\ &\operatorname{rank}\mathbf{W}(X,Z)\big|_{\mathfrak{T}_X(w)} = 2 \end{aligned}$$

as well as

$$\mathbf{W}(X,Z)\circ Z_\sigma = Z_\sigma \quad \text{for all } \sigma = 1,2,\ldots,n;$$

(W3) $\mathbf{W}(X,Z)$ is positive definite, i.e. with a real constant $\omega_0 \in [0,+\infty)$ it holds

$$(1+\omega_0)^{-1}|\xi|^2 \le \xi\circ\mathbf{W}(X,Z)\circ\xi \le (1+\omega_0)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{n+2};$$

(W4) $\mathbf{W}(X,Z)$ is normalized in the following sense

$$\mathbf{W}(X,Z) = 1 \quad \text{for all } X,Z.$$

Weight matrices were first introduced in Sauvigny [141] for a new representation of critical points for anisotropic variational problems in $\mathbb{R}^3$ in parametric form.

Sauvigny's methods were further developed and applied e.g. in Bergner und Dittrich [12], Clarenz [30], [31], Clarenz and von der Mosel [32], Fröhlich [62], [64], and Winklmann [172], [173] to inhomogeneous and anisotropic variational problems. In chapter 11 we will present some of those results.

Definition 4.1. A matrix $\mathbf{W}(X,Z)$ with the properties (W1) to (W4) is called a *weight matrix.*

By means of such a weight matrix we are able to create a differential geometry of Finslerian type. For this purpose we start with the

Definition 4.2. The *weighted first fundamental form* $I_W(X) \in \mathbb{R}^2 \times \mathbb{R}^2$ of the immersion $X\colon B \to \mathbb{R}^{n+2}$ with ONF N is defined as

$$I_W(X) = (h_{ij})_{i,j=1,2}\,, \quad h_{ij} := X_{u^i} \circ \mathbf{W}(X,N) \circ X_{u^j}\,. \tag{4.2}$$

Note that $h_{ij} \equiv g_{ij}$ with the coefficients g_{ij} of the classical first fundamental form if $\mathbf{W}(X,Z) \equiv \mathbb{E}^{n+2}$ with the $(n+2)$-dimensional unit matrix $\mathbb{E}^{n+2}$.

We furthermore remark that the weighted first fundamental $I_W(X)$ of an immersion $X\colon B \to \mathbb{R}^{n+2}$ is symmetric and positive definite since the matrix $\mathbf{W}(X,Z)$ is then symmetric and positive definite.

Definition 4.3. The *weighted second* and the *weighted third fundamental forms* of the immersion $X\colon B \to \mathbb{R}^{n+2}$ are defined as

$$\begin{aligned} I\!I_{W,N_\sigma}(X) &= (L_{\sigma,ij})_{i,j=1,2}\,, \quad L_{\sigma,ij} := -X_{u^i} \cdot N_{\sigma,u^j} = X_{u^iu^j} \cdot N_\sigma\,, \\ I\!I\!I_{W,N_\sigma}(X) &= (f_{\sigma,ij})_{i,j=1,2}\,, \quad f_{N_\sigma,ij} := N_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N_{\sigma,u^j}\,. \end{aligned} \tag{4.3}$$

with a unit normal vector $N_\sigma \in \mathbb{R}^{n+2}$ of an ONF N.

With these definitions of the weighted second and third fundamental forms we essentially follow Sauvigny [141] and Fröhlich [64]. Alternative ways were encouraged by Clarenz [31] in case $n = 1$ and by Winklmann [172], [173] for m-dimensional manifolds in $\mathbb{R}^{m+1}$.

Consider that, in contrast to the case $n = 1$, the vector N_{σ,u^i} *is not necessarily tangential* due to the presence of torsion coefficients. A further definition thus also commonly used only involves the tangential parts of N_{σ,u^i}.

To illustrate this fact consider an unit normal vector $N_\sigma = N$ with its derivatives splitted up into tangential and normal parts

$$N_{\sigma,u^i} = N^\top_{\sigma,u^i} + N^\perp_{\sigma,u^i}\,,$$

the latter vanishing identically if $n = 1$. We compute

$$\begin{aligned} &N_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N_{\sigma,u^j} \\ &\quad = (N^\perp_{\sigma,u^i} + N^\top_{\sigma,u^i}) \circ \mathbf{W}(X,N)^{-1} \circ (N^\perp_{\sigma,u^j} + N^\top_{\sigma,u^j}) \\ &\quad = N^\perp_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\perp_{\sigma,u^j} + N^\perp_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\top_{\sigma,u^j} \\ &\qquad + N^\top_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\perp_{\sigma,u^j} + N^\top_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\top_{\sigma,u^j} \\ &\quad = N^\top_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\top_{\sigma,u^j} + N^\perp_{\sigma,u^i} \cdot N^\perp_{\sigma,u^j} \end{aligned}$$

due to $\mathbf{W}(X,N)^{-1} \circ N^\top_{\sigma,u^j} = N^\top_{\sigma,u^j}$ as well as

$$N^\perp_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\top_{\sigma,u^j} = 0, \quad N^\top_{\sigma,u^i} \circ \mathbf{W}(X,N)^{-1} \circ N^\perp_{\sigma,u^j} = 0.$$

In particular, if the ONF N is free of torsion then

$$\begin{aligned} &\mathit{III}_{W,N_\sigma}(X) \equiv \widehat{\mathit{III}}_{W,N_\sigma}(X) = (\widehat{f}_{\sigma,ij})_{i,j=1,2} \\ &\text{with} \quad \widehat{f}_{\sigma,ij} := N_{u^i}^\top \circ \mathbf{W}(X,N)^{-1} \circ N_{\sigma,u^j}^\top \,. \end{aligned} \tag{4.4}$$

It is the form $\widehat{\mathit{III}}_W(X)$ which is often introduced as a third fundamental form, see e.g. Brauner's textbook [18] for an elaboration of the non-weighted case.

Finally we want to remark that property (W4) particularly ensures that the classical area element W equals its weighted counter-part W_W in the following sense

$$W_W := \det\left(DX^T \circ \mathbf{W}(X,N) \circ DX\right) = \det\left(DX^T \circ DX\right) = W \tag{4.5}$$

with the Jacobian DX operating on the tangent plane of the surface.

Thus we do not need to distinguish between a "weighted" and a "non-weighted" area element. The same holds true for the Gaussian curvature, but as we will see later, it is necessary to introduce a *weighted mean curvature.*

4.3 Differential equations

The equations of Gauss and Weingarten reflect the way of representation of the derivatives $X_{u^iu^j}$ and N_{σ,u^i} in terms of a moving frame $\{X_u, X_v, N_1, \ldots, N_n\}$.

In chapter 2 we derived these equations for the case $\mathbf{W}(X,Z) \equiv \mathbb{E}^{n+2}$. Now we want to present its counterparts in the weighted setup, but we will omit the proofs which actually work as in the classical case.

Let that a weight matrix $\mathbf{W}(X,Z) \in \mathbb{R}^{(n+2)\times(n+2)}$ be given.

Theorem 4.1. *(Gauss equations in weighted form)*
Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then it holds

$$X_{u^iu^j} = \sum_{k=1}^{2} (\Gamma_{ij}^k + \Omega_{ij}^k) X_{u^k} + \sum_{\vartheta=1}^{n} L_{\vartheta,ij} N_\vartheta \tag{4.6}$$

for all $i,j = 1,2$, with the Christoffel symbols

$$\Gamma_{ij}^k := \sum_{\ell=1}^{2} \frac{1}{2} h^{k\ell}(h_{\ell i,u^j} + h_{j\ell,u^i} - h_{ij,u^\ell}) \tag{4.7}$$

and the corrective terms

$$\Omega_{ij}^k := -\frac{1}{2} \sum_{\ell=1}^{2} h^{k\ell}(\omega_{\ell ij} + \omega_{j\ell i} - \omega_{ij\ell}), \quad \omega_{ij\ell} := X_{u^i} \circ \mathbf{W}(X,N)_{u^\ell} \circ X_{u^j} \,. \tag{4.8}$$

We remark

$$\mathbf{W}(X,N)_{u^\ell} = \mathbf{W}_X(X,N)\circ X_{u^\ell} + \mathbf{W}_Z(X,N)\circ N_{u^\ell}$$

with the settings

$$\mathbf{W}_X(X,N)\circ X_{u^\ell} = \left(\sum_{k=1}^{n+2} w_{ij,x^k} x^k_{u^\ell}\right)_{i,j=1,\ldots,n+2} \in \mathbb{R}^{(n+2)\times(n+2)} \quad \text{etc.}$$

Theorem 4.2. *(Weingarten equations in weighted form)*
Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ together with an ONF N be given. Then it holds

$$N_{\sigma,u^i} = -\sum_{j,k=1}^{2} L_{\sigma,ij}h^{jk}\,\mathbf{W}(X,N)\circ X_{u^k} + \sum_{\vartheta=1}^{n} T_{\sigma,i}^{\vartheta} N_\vartheta \tag{4.9}$$

for all $i=1,2$, $\sigma=1,\ldots,n$.

4.4 The weighted mean curvature system

Due to the positive definiteness and symmetry of $\mathbf{W}(X,Z)$, the line element

$$ds_W^2 = \sum_{i,j=1}^{2} h_{ij}\,du^i du^j$$

is of Riemannian type. Thus we may introduce *weighted conformal parameters* $(u,v)\in B$ satisfying

$$\begin{aligned} &X_u\circ\mathbf{W}(X,N)\circ X_v = W = X_v\circ\mathbf{W}(X,N)\circ X_v\,,\\ &X_u\circ\mathbf{W}(X,N)\circ X_v = 0 \quad \text{in } B \end{aligned} \tag{4.10}$$

with the area element W since $W_W = W$ following our discussion above. We also refer to Sauvigny [143], chapter XII, §8.

Then the Christoffel symbols Γ_{ij}^k from (4.7) w.r.t. the coefficients h_{ij} satisfy again the relations (2.6). In particular, we infer

$$\Gamma_{11}^1+\Gamma_{22}^1=0,\quad \Gamma_{11}^2+\Gamma_{22}^2=0.$$

Now we can evaluate the Gauss equations (4.6) to get the mean curvature system in weighted form generalizing the classical system

$$\Delta X = 2HW \quad \text{in } B$$

for the conformally parametrized immersion X.

Corollary 4.1. *Let the weighted conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$ together with an ONF N be given. Then it holds*

$$\begin{aligned}\Delta X &= (\Omega_{11}^1+\Omega_{22}^1)X_u+(\Omega_{11}^2+\Omega_{22}^2)X_v+2\sum_{\vartheta=1}^{n}H_{W,\vartheta}WN_\vartheta \\ &= (\Omega_{11}^1+\Omega_{22}^1)X_u+(\Omega_{11}^2+\Omega_{22}^2)X_v+2H_WW\end{aligned} \tag{4.11}$$

with the weighted mean curvature vector

$$H_W := \sum_{\vartheta=1}^{n}H_{W,\vartheta}N_\vartheta\,, \quad H_{W,\vartheta} := \sum_{i,j=1}^{2}h^{ij}L_{ij}\,. \tag{4.12}$$

Note that (4.11) is a coupled elliptic system for the surface vector X and the unit normal vectors N_σ of some ONF N and their derivatives.

Definition 4.4. The immersion $X\colon B\to\mathbb{R}^{n+2}$ is called a *weighted minimal surface* w.r.t to the weight matrix $\mathbf{W}(X,Z)$ if it holds

$$H_W\equiv 0 \quad \text{in } B.$$

Thus minimal surfaces with the property $H\equiv 0$ are special weighted minimal surfaces in the classical case $\mathbf{W}(X,Z)\equiv\mathbb{E}^{n+2}$.

Definition 4.5. We define the *weighted Laplacian* Δ_W w.r.t. a weight matrix $\mathbf{W}(X,Z)$ as the elliptic operator

$$\Delta_W := \Delta-(\Omega_{11}^1+\Omega_{22}^1)\frac{\partial}{\partial u}-(\Omega_{11}^2+\Omega_{22}^2)\frac{\partial}{\partial v} \tag{4.13}$$

with the Euclidean Laplacian Δ.

Then the weighted conformally parametrized immersion X satisfies

$$\Delta_W X = 2H_WW$$

in analogy to the classical mean curvature system $\Delta X = 2HW$.

Remark 4.1. Clarenz and von der Mosel in [32] proposed another Laplace-type operator to analyse critical points for anisotropic and inhomogeneous variational problems, namely

$$\Delta_C := \frac{1}{W}\sum_{i=1}^{2}\frac{\partial}{\partial u^i}\left(\sum_{j,k,\ell=1}^{2}g^{ik}h_{k\ell}g^{j\ell}\frac{\partial}{\partial u^j}\right); \tag{4.14}$$

see also the references in [32]. Their formalism was picked up e.g. by Cluttberbuck [33] and Bergner and Dittrich [12].

4.5 The Codazzi-Mainardi equations

We want to evaluate the integrability conditions which particularly pour out from

$$\partial_{uv} X_{u^i} - \partial_{vu} X_{u^i} = 0 \quad \text{for } i = 1,2,$$

by comparing the respective coefficients of the normal directions. For this purpose we write explicitely

$$\begin{aligned} X_{u^i u} &= (\Gamma_{i1}^1 + \Omega_{i1}^1) X_u + (\Gamma_{i1}^2 + \Omega_{i1}^2) X_v + \sum_{\sigma=1}^n L_{\sigma,i1} N_\sigma \,, \\ X_{u^i v} &= (\Gamma_{i2}^1 + \Omega_{i2}^1) X_u + (\Gamma_{i2}^2 + \Omega_{i2}^2) X_v + \sum_{\sigma=1}^n L_{\sigma,i2} N_\sigma \end{aligned}$$

for $i = 1,2$. A second differentiation gives

$$\begin{aligned} \partial_{uv} X_{u^i} = &\Big\{ \Gamma_{i1,v}^1 + \Omega_{i1,v}^1 + (\Gamma_{i1}^1 + \Omega_{i1}^1)(\Gamma_{12}^1 + \Omega_{12}^1) + (\Gamma_{i1}^2 + \Omega_{i1}^2)(\Gamma_{22}^1 + \Omega_{22}^1) \Big\} X_u \\ &+ \Big\{ \Gamma_{i1,v}^2 + \Omega_{i1,v}^2 + (\Gamma_{i1}^1 + \Omega_{i1}^1)(\Gamma_{12}^2 + \Omega_{12}^2) + (\Gamma_{i1}^2 + \Omega_{i1}^2)(\Gamma_{22}^2 + \Omega_{22}^2) \Big\} X_v \\ &+ \sum_{\sigma=1}^n \left\{ L_{\sigma,i1,v} + (\Gamma_{i1}^1 + \Omega_{i1}^1) L_{\sigma,12} + (\Gamma_{i1}^2 + \Omega_{i1}^2) L_{\sigma,22} + \sum_{\omega=1}^n L_{\omega,i1} T_{\omega,2}^\sigma \right\} N_\sigma \\ &- \sum_{\ell,m=1}^2 \sum_{\sigma=1}^n L_{\sigma,i1} L_{\sigma,2\ell} h^{\ell m} \mathbf{W}(X,N) \circ X_{u^m} \end{aligned}$$

as well as, interchanging u and v,

$$\begin{aligned} \partial_{vu} X_{u^i} = &\Big\{ \Gamma_{i2,u}^1 + \Omega_{i2,u}^1 + (\Gamma_{i2}^1 + \Omega_{i2}^1)(\Gamma_{11}^1 + \Omega_{11}^1) + (\Gamma_{i2}^2 + \Omega_{i2}^2)(\Gamma_{12}^1 + \Omega_{12}^1) \Big\} X_u \\ &+ \Big\{ \Gamma_{i2,u}^2 + \Omega_{i2,u}^2 + (\Gamma_{i2}^1 + \Omega_{i2}^1)(\Gamma_{11}^2 + \Omega_{11}^2) + (\Gamma_{i2}^2 + \Omega_{i2}^2)(\Gamma_{12}^2 + \Omega_{12}^2) \Big\} X_v \\ &+ \sum_{\sigma=1}^n \left\{ L_{\sigma,i2,u} + (\Gamma_{i2}^1 + \Omega_{i2}^1) L_{\sigma,11} + (\Gamma_{i2}^2 + \Omega_{i2}^2) L_{\sigma,12} + \sum_{\omega=1}^n L_{\omega,i2} T_{\omega,1}^\sigma \right\} N_\sigma \\ &- \sum_{\ell,m=1}^2 \sum_{\sigma=1}^n L_{\sigma,i2} L_{\sigma,1\ell} h^{\ell m} \mathbf{W}(X,N) \circ X_{u^m} \,. \end{aligned}$$

Focusing on the normal components of both identitites proves the following version of the Codazzi-Mainardi equations which completes this section.

Theorem 4.3. *(Codazzi-Mainardi integrability conditions in weighted form)*
Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ *together with ONF* N *be given. Then there hold*

$$\begin{aligned} &L_{\sigma,i1,v} + (\Gamma_{i1}^1 + \Omega_{i1}^1)L_{\sigma,12} + (\Gamma_{i1}^2 + \Omega_{i1}^2)L_{\sigma,22} + \sum_{\omega=1}^{n} L_{\omega,i1}T_{\omega,2}^{\sigma} \\ &\quad = L_{\sigma,i2,u} + (\Gamma_{i2}^1 + \Omega_{i2}^1)L_{\sigma,11} + (\Gamma_{i2}^2 + \Omega_{i2}^2)L_{\sigma,12} + \sum_{\omega=1}^{n} L_{\omega,i2}T_{\omega,1}^{\sigma} \end{aligned} \tag{4.15}$$

for all $i = 1,2$ *and* $\sigma = 1,2,\ldots,n$.

¡ These equations differ from the classical integrability conditions (3.6) in the appearence of the additional data Ω_{ij}^k. We will make essential use them in the next chapter.

Chapter 5
The Hopf vector

We prove a linear connection between the three weighted fundamental forms what enables us to compare the spherical energy of an immersion with its curvatures and torsions. Furthermore we consider complex-valued Hopf fields and verify their analyticity in the sense of Bers and Vekua. This allows us to characterize singular points of the spherical mapping of special surfaces.

5.1 Linear dependence of the weighted fundamental forms

The three weighted fundamental forms can not be chosen independently from each other as the following result shows.

Theorem 5.1. *(Linear dependence of the fundamental forms)*
Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ together with an ONF N be given. Then it holds

$$\mathit{III}_{W,N_\sigma}(X)-2H_{W,\sigma}\,\mathit{II}_{W,\sigma}(X)+K_\sigma\,I_W(X)=\left(\sum_{\vartheta=1}^{n}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}\right)_{i,j=1,2} \tag{5.1}$$

for $\sigma=1,\ldots,n$ with the Gauss curvatures K_σ from (2.9) and the weighted mean curvatures $H_{W,\sigma}$ from (4.12).

Proof. We evaluate the coefficients $f_{\sigma,ij}=N_{\sigma,u^i}\cdot N_{\sigma,u^j}$ of the weighted third fundamental form to get

$$\begin{aligned}
f_{\sigma,ij}&=\left(-\sum_{m,n=1}^{2}L_{\sigma,im}h^{mn}\,\mathbf{W}(X,N)\circ X_{u^n}+\sum_{\vartheta=1}^{n}T_{\sigma,i}^{\vartheta}N_\vartheta\right)\circ\mathbf{W}(X,N)^{-1}\circ\ldots\\
&\qquad\ldots\circ\left(-\sum_{r,s=1}^{2}L_{\sigma,jr}h^{rs}\,\mathbf{W}(X,N)\circ X_{u^s}+\sum_{\lambda=1}^{n}T_{\sigma,j}^{\lambda}N_\lambda\right)\\
&=\sum_{m,n,r,s=1}^{2}L_{\sigma,im}L_{\sigma,jr}h^{mn}h^{rs}h_{ns}+\sum_{\vartheta,\lambda=1}^{n}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\lambda}\delta_{\vartheta\lambda}\\
&=\sum_{m,r=1}^{2}L_{\sigma,im}L_{\sigma,jr}h^{mr}+\sum_{\vartheta=1}^{n}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}\,.
\end{aligned}$$

Now we exemplarily compute for $i,j=1$

$$\begin{aligned} f_{\sigma,11}-\sum_{\vartheta=1}^{n}(T_{\sigma,1}^{\vartheta})^2 &= L_{\sigma,11}^2h^{11}+2L_{\sigma,11}L_{\sigma,12}h^{12}+L_{\sigma,12}^2h^{22} \\ &= \left\{L_{\sigma,11}h^{11}+2L_{\sigma,12}h^{12}+L_{\sigma,22}h^{22}\right\}L_{\sigma,11} \\ &\quad -(L_{\sigma,11}L_{\sigma,22}-L_{\sigma,12}^2)h^{22} \\ &= 2H_{W,\sigma}L_{\sigma,11}-K_\sigma h_{11}\,. \end{aligned}$$

The remaining coefficients follow analogously. □

The non-weighted version of this identity runs

$$I\!I\!I_{N_\sigma}(X)-2H_\sigma I\!I_{N_\sigma}(X)+K_\sigma I(X)=\left(\sum_{\vartheta=1}^{n}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}\right)_{i,j=1,2}.$$

Finally, using the coefficients $\widehat{f}_{\sigma,ij}=N_{\sigma,u^i}^{\top}\circ\mathbf{W}(X,N)^{-1}\circ N_{\sigma,u^j}^{\top}$ from (4.4) we would arrive at the relation

$$\widehat{I\!I\!I}_{W,N_\sigma}(X)-2H_{W,\sigma}I\!I_{W,N_\sigma}(X)+K_\sigma I_W(X)=\mathbf{0} \tag{5.2}$$

where we take the identity $L_{N,ij}=-X_{u^j}\cdot N_{u^j}=-X_{u^i}\cdot N_{u^j}^{\top}$ into account.

5.2 Minimal surfaces and weighted minimal surfaces

Consider a conformally parametrized minimal surface $X\colon B\longrightarrow\mathbb{R}^3$ with the characteristic property

$$H\equiv 0$$

for its scalar mean curvature H. Then it holds

$$I\!I\!I(X)-KW\mathbb{E}^2=\mathbf{0}$$

with the twodimensional unit matrix $\mathbb{E}^2\subset\mathbb{R}^{2\times 2}$. Since $I(X)$ is given in diagonal form we infer *that the spherical mapping* $N\colon B\longrightarrow\mathbb{R}^3$ *is also conformally parametrized.*

The same is true for weighted minimal surfaces satisfying

$$H_W\equiv 0$$

as well as the relations (4.10) w.r.t. some weight matrix $\mathbf{W}(X,Z)$. Namely, we infer

$$I\!I\!I_W(X)-KW\mathbb{E}^2=\mathbf{0}.$$

5.3 The torsion of orthonormal normal frames

Let us consider again the non-weighted case $\mathbf{W}(X,\mathfrak{z}) \equiv \mathbb{E}^{n+2}$. We write the linear dependence of the fundamental forms from (5.1) as

$$e_{\sigma,ij} - 2H_\sigma L_{\sigma,ij} + K_\sigma g_{ij} = \sum_{\vartheta=1}^{n} T_{\sigma,i}^{\vartheta} T_{\sigma,j}^{\vartheta}$$

with the coefficients $e_{\sigma,ij} = N_{\sigma,u^i} \cdot N_{\sigma,u^j}$.

Corollary 5.1. *Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then*

$$\begin{aligned} e_{\sigma,11} &= 2H_\sigma L_{\sigma,11} - K_\sigma W + \sum_{\vartheta=1}^{n} (T_{\sigma,1}^{\vartheta})^2, \\ e_{\sigma,12} &= 2H_\sigma L_{\sigma,12} + \sum_{\vartheta=1}^{n} T_{\sigma,1}^{\vartheta} T_{\sigma,2}^{\vartheta}, \\ e_{\sigma,22} &= 2H_\sigma L_{\sigma,22} - K_\sigma W + \sum_{\vartheta=1}^{n} (T_{\sigma,2}^{\vartheta})^2. \end{aligned}$$

In particular, we infer

$$|\nabla N_\sigma|^2 = e_{\sigma,11} + e_{\sigma,22} = 2(2H_\sigma^2 - K_\sigma)W + \sum_{\vartheta=1}^{n} \left\{ (T_{\sigma,1}^{\vartheta})^2 + (T_{\sigma,2}^{\vartheta})^2 \right\} \tag{5.3}$$

for the square of the Euclidean gradient $\nabla N_\sigma = (N_{\sigma,u}, N_{\sigma,v})$.

The sum on the right hand side of the latter identity motivates us for the following definition.

Definition 5.1. The *torsion of the ONF N* is defined as

$$T := \frac{1}{2} \sum_{i,j=1}^{2} \sum_{\sigma,\vartheta=1}^{n} g^{ij} T_{\sigma,i}^{\vartheta} T_{\sigma,j}^{\vartheta}. \tag{5.4}$$

Remark 5.1. A torsion within our Finsler-type setting could take the form

$$T_W := \frac{1}{2} \sum_{i,j=1}^{2} \sum_{\sigma,\vartheta=1}^{n} h^{ij} T_{\sigma,i}^{\vartheta} T_{\sigma,j}^{\vartheta} \tag{5.5}$$

Note here that for an arbitrary vector $z \in \mathbb{R}^2$ it holds

$$\begin{aligned} z \circ (h_{ij})_{i,j=1,2} \circ z &= z \circ (DX \circ \mathbf{W}(X,N) \circ DX) \circ z \\ &= (z \circ DX) \circ \mathbf{W}(X,N) \circ (DX \circ z). \end{aligned}$$

Now apply property (W3), and we conclude that T and T_W are equivalent in the following sense

$$\frac{1}{1+\omega_0}\,T \le T_W \le (1+\omega_0)T. \tag{5.6}$$

While the torsion T (and so T_W) depends on the chosen ONF N, it is invariant w.r.t. parameter transformations of class $\mathfrak{P}$.

Proposition 5.1. *T is invariant w.r.t. parameter transformations of class $\mathfrak{P}$.*

Proof. Using the results from section 1.7 we calculate

$$\begin{aligned}
\sum_{i,j=1}^{2}\sum_{\sigma,\vartheta=1}^{n} g^{ij}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta} &= \sum_{i,j=1}^{2}\sum_{\alpha,\beta\mu,\nu=1}^{2}\sum_{\sigma,\vartheta=1}^{n} \overline{\Lambda}_{\mu}^{i}\overline{\Lambda}_{\nu}^{j}\Lambda_{i}^{\alpha}\Lambda_{j}^{\beta}g^{\mu\nu}T_{\sigma,\alpha}^{\vartheta}T_{\sigma,\beta}^{\vartheta} \\
&= \sum_{\alpha,\beta,\mu,\nu=1}^{2}\sum_{\sigma,\vartheta=1}^{n} \delta_{\mu}^{\alpha}\delta_{\nu}^{\beta}g^{\mu\nu}T_{\sigma,\alpha}^{\vartheta}T_{\sigma,\beta}^{\vartheta} \\
&= \sum_{\alpha,\beta=1}^{2}\sum_{\sigma,\vartheta=1}^{n} g^{\alpha\beta}T_{\sigma,\alpha}^{\vartheta}T_{\sigma,\beta}^{\vartheta}
\end{aligned}$$

proving the claim. □

Now taking the definition of H and K from (2.5) and (2.9) into account, a direct calculation proves the

Theorem 5.2. *(Energy density of the Gauss map)*
Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then it holds

$$\sum_{\sigma=1}^{n} |\nabla N_\sigma|^2 = 2(2H^2-K)W + 2TW \tag{5.7}$$

with the squared length H^2 of the mean curvature vector H, the Gauss curvature K and the torsion T of the ONF N.

This identity is very useful for various energy and gradient estimates as we will see later. For the moment we want to mention that an integration yields

$$\sum_{\sigma=1}^{n}\iint\limits_{B} |\nabla N_\sigma|^2\,dudv = 2\iint\limits_{B}(2H^2-K)W\,dudv + 2\iint\limits_{B} TW\,dudv$$

what leads us directly to our next concept.

5.4 The functional of total torsion

Definition 5.2. Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then the functional

$$\mathscr{T}_X[N] := \iint_B TW\,dudv$$

is called the *total torsion* of N.

Proposition 5.2. *The functional $\mathscr{T}_X[N]$ of total torsion is invariant w.r.t. to parameter transformations of class $\mathfrak{P}$.*

Proof. This follows immediately from Proposition 5.1.

Though $\mathscr{T}_X[N]$ is independent of the choice of the parametrization, *it depends on the choice of the ONF N*. In chapters 7 and 8 below we will present methods to establish existence and regularity of orthonormal normal frames critical for the functional of total torsion.

5.5 The curvatura integra

Definition 5.3. Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ be given. Then the functional

$$\mathscr{K}[X] := \iint_B KW\,dudv$$

is called its *curvatura integra.*

We want to omit proving its independence of the parametrization as well as the choice of the ONF. Both facts reliy strongly on Proposition 2.3 from chapter 2.

Proposition 5.3. *The curvatura integra $\mathscr{K}[X]$ is invariant w.r.t. to parameter transformations of class $\mathfrak{P}$, and it does not depend on the choice of the ONF N.*

Now we want to prove the famous *theorem of Bonnet and Gauss* which relates the curvatura integra of an immersion with the geodesic curvature κ_g of its boundary curve.

The curvature κ_g is roughly defined as follows: Consider a point P of the surface's boundary curve $\Gamma \subset \mathbb{R}^{n+2}$ and the tangential plane of the surface at P. Project Γ locally around P onto this tangential plane to get a new planar curve. Then $\kappa_g(P)$ is defined as the usual curvature of the projected planar curve at P.

For proving the integral formula of Bonnet and Gauss we need the representation

$$K = -\frac{1}{W}\,\Delta\log\sqrt{W}$$

of the Gauss curvature using conformal parameters which we derived in chapter 3.

Theorem 5.3. *(Integral formula of Bonnet and Gauss)*
Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ be given. Then it holds

$$\iint\limits_B KW\,dudv = 2\pi-\int\limits_0^{2\pi}\kappa_g(s)\,ds.$$

Proof. We follow Dierkes et al. [44]. Suppose $X=X(u,v)$ is given in conformal parameters $(u,v)\in B$. Let $\nu\in\mathbb{R}^2$ denote the outer unit normal vector at the boundary curve ∂B. Let this curve be given in parametric form $(\cos t,\sin t)$, $t\in[0,2\pi)$. Now we take *Minding's formula*

$$\kappa_g(t)\sqrt{W(t)}=1+\frac{\partial}{\partial\nu}\log\sqrt{W(t)}\quad\text{for all } t\in[0,2\pi)$$

into account which remains unproved here (see [44], chapter 1). Partial integration then yields

$$\begin{aligned}-\iint\limits_B KW\,dudv &= \iint\limits_B \Delta\log\sqrt{W(t)}\,dudv = \int\limits_0^{2\pi}\frac{\partial}{\partial\nu}\log\sqrt{W(t)}\,dt\\ &= \int\limits_0^{2\pi}\left\{\kappa_g(t)\sqrt{W(t)}-1\right\}dt = \int\limits_{\partial B}\kappa_g(s)\,ds-2\pi\end{aligned}$$

proving the statement. □

5.6 Hopf functions and Hopf vector

Consider an immersion $X\colon B\to\mathbb{R}^3$ with second fundamental form $(L_{ij})_{i,j=1,2}$.

In 1950, H. Hopf [94] (see also Jost's textbook [102]) discovered that the complex-valued function

$$\mathscr{H}(w):=L_{11}(w)-L_{22}(w)-2iL_{12}(w),\quad w\in B,$$

is *holomorphic if the scalar mean curvature H of the surface is constant.*[1]

In this section we introduce a generalized vector-valued Hopf function adapted to our Finsler-type setting. We will make essential use of complex-analytical tools of Bers [14] and Vekua [160], [161].

Let a weight matrix $\mathbf{W}(X,Z)\in\mathbb{R}^{(n+2)\times(n+2)}$ be given.

[1] In particular, this fact turns out to be a key ingredient for proving that a compact embedding of constant, nonvanishing mean curvature is indeed the standard sphere.

Lemma 5.1. *Let the weighted conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$ together with an ONF N be given. Define the functions*

$$a:=\Omega_{22}^1+\Omega_{21}^2\,,\quad b:=\Omega_{22}^2-\Omega_{21}^1\,,\quad c:=\Omega_{12}^1+\Omega_{11}^2\,,\quad d:=\Omega_{12}^2-\Omega_{11}^1\,,\tag{5.8}$$

with Ω_{ij}^k being the coefficients from Gauss equations in weighted form from section 4.3, as well as

$$\begin{aligned}r_\sigma &:= W_uH_{W,\sigma}+2W\partial_uH_{W,\sigma}-2\Omega_{21}^2WH_{W,\sigma}+\sum_{\vartheta=1}^n(L_{\vartheta,22}T_{\vartheta,1}^\sigma-L_{\vartheta,21}T_{\vartheta,2}^\sigma)\,,\\ s_\sigma &:= W_vH_{W,\sigma}-2\Omega_{11}^2WH_{W,\sigma}+\sum_{\vartheta=1}^n(L_{\vartheta,12}T_{\vartheta,1}^\sigma-L_{\vartheta,11}T_{\vartheta,2}^\sigma)\end{aligned}\tag{5.9}$$

with the weighted mean curvature vector H_W from section 4.4. Then there hold

$$\begin{aligned}L_{\sigma,11,u}+L_{\sigma,12,v} &= aL_{\sigma,11}+bL_{\sigma,12}+r_\sigma\,,\\ L_{\sigma,11,v}-L_{\sigma,12,u} &= cL_{\sigma,11}+dL_{\sigma,12}+s_\sigma\end{aligned}\tag{5.10}$$

for $\sigma=1,\ldots,n$.

Proof. First come the Codazzi-Mainardi equations from section 4.5, i.e.

$$\begin{aligned}&L_{\sigma,11,v}+(\Gamma_{11}^1+\Omega_{11}^1)L_{\sigma,12}+(\Gamma_{11}^2+\Omega_{11}^2)L_{\sigma,22}+\sum_{\vartheta=1}^nL_{\vartheta,11}T_{\vartheta,2}^\sigma\\ &\quad=L_{\sigma,12,u}+(\Gamma_{12}^1+\Omega_{12}^1)L_{\sigma,11}+(\Gamma_{12}^2+\Omega_{12}^2)L_{\sigma,12}+\sum_{\vartheta=1}^nL_{\vartheta,12}T_{\vartheta,1}^\sigma\,,\\ &L_{\sigma,21,v}+(\Gamma_{21}^1+\Omega_{21}^1)L_{\sigma,12}+(\Gamma_{21}^2+\Omega_{21}^2)L_{\sigma,22}+\sum_{\vartheta=1}^nL_{\vartheta,21}T_{\vartheta,2}^\sigma\\ &\quad=L_{\sigma,22,u}+(\Gamma_{22}^1+\Omega_{22}^1)L_{\sigma,11}+(\Gamma_{22}^2+\Omega_{22}^2)L_{\sigma,12}+\sum_{\vartheta=1}^nL_{\vartheta,22}T_{\vartheta,1}^\sigma\,.\end{aligned}$$

Thus, using weighted conformal parameters together with (4.10) and (2.6) we get

$$\begin{aligned}&L_{\sigma,11,v}-\frac{W_v}{2W}L_{\sigma,22}+\sum_{m=1}^2\Omega_{11}^mL_{\sigma,m2}+\sum_{\vartheta=1}^nL_{\vartheta,11}T_{\vartheta,2}^\sigma\\ &\quad=L_{\sigma,12,u}+\frac{W_v}{2W}L_{\sigma,11}+\sum_{m=1}^2\Omega_{12}^mL_{\sigma,m1}+\sum_{\vartheta=1}^nL_{\vartheta,12}T_{\vartheta,1}^\sigma\,,\\ &L_{\sigma,12,v}+\frac{W_u}{2W}L_{\sigma,22}+\sum_{m=1}^2\Omega_{21}^mL_{\sigma,m2}+\sum_{\vartheta=1}^nL_{\vartheta,21}T_{\vartheta,2}^\sigma\\ &\quad=L_{\sigma,22,u}-\frac{W_u}{2W}L_{\sigma,11}+\sum_{m=1}^2\Omega_{22}^mL_{\sigma,m1}+\sum_{\vartheta=1}^nL_{\vartheta,22}T_{\vartheta,1}^\sigma\,.\end{aligned}$$

Rearranging gives the identities

$$
\begin{aligned}
&L_{\sigma,11,v} - L_{\sigma,12,u} \\
&\qquad = (\Omega^1_{12} + \Omega^2_{11})L_{\sigma,11} + (\Omega^2_{12} - \Omega^1_{11})L_{\sigma,12} - 2\Omega^2_{11} W H_{W,\sigma} + W_v H_{W,\sigma} \\
&\qquad\quad + \sum_{\vartheta=1}^{n} (L_{\vartheta,12} T^{\sigma}_{\vartheta,1} - L_{\vartheta,11} T^{\sigma}_{\vartheta,2}), \\
&L_{\sigma,11,u} + L_{\sigma,12,v} \\
&\qquad = (\Omega^1_{22} + \Omega^2_{21})L_{\sigma,11} + (\Omega^2_{22} - \Omega^1_{21})L_{\sigma,12} - 2\Omega^2_{21} W H_{W,\sigma} + W_u H_{W,\sigma} \\
&\qquad\quad + 2W\partial_u H_{W,\sigma} + \sum_{\vartheta=1}^{n} (L_{\vartheta,22} T^{\sigma}_{\vartheta,1} - L_{\vartheta,21} T^{\sigma}_{\vartheta,2})
\end{aligned}
$$

for $\sigma = 1, \ldots, n$. This proves the statement. □

These identities enable us to prove generalized analyticity of the following complex-valued functions.

Definition 5.4. The *Hopf function* $\mathscr{H}_N \in \mathbb{C}$ of an immersion X *w.r.t. some unit normal vector* N is defined as

$$
\mathscr{H}_N(w) := L_{N,11}(w) - L_{N,22}(w) - 2iL_{N,12}(w), \quad w \in B. \tag{5.11}
$$

Note that it holds

$$
\mathscr{H}_N = 2L_{N,11} - (L_{N,11} + L_{N,22}) - 2iL_{N,12} = -2H_{W,N}W + 2L_{N,11} - 2iL_{N,12}\,.
$$

Next we introduce the so-called *Wirtinger symbols* (see e.g. Vekua [161]): Let $\Phi\colon \mathbb{C} \to \mathbb{C}^n$ be continuously differentiable. Then we define its *complex derivatives*

$$
\Phi_w \equiv \partial_w \Phi := \frac{1}{2}\,(\Phi_u - i\Phi_v), \quad \Phi_{\overline{w}} \equiv \partial_{\overline{w}} := \frac{1}{2}\,(\Phi_u + i\Phi_v).
$$

Lemma 5.2. *Let the weighted conformally parametrized immersion* $X\colon B \to \mathbb{R}^{n+2}$ *together with an ONF* N *be given. Then it holds*

$$
\begin{aligned}
\partial_{\overline{w}} \mathscr{H}_\sigma &= A\mathscr{H}_\sigma + B\overline{\mathscr{H}}_\sigma + 2(A + B - \Omega^2_{21} - i\Omega^2_{11})H_{W,\sigma}W + 2\partial_w H_{W,\sigma}W \\
&\quad + \sum_{\vartheta=1}^{n} \Big\{ (L_{\vartheta,22} + iL_{\vartheta,12})T^{\sigma}_{\vartheta,1} - (L_{\vartheta,21} + iL_{\vartheta,11})T^{\sigma}_{\vartheta,2} \Big\}
\end{aligned}
$$

for $\sigma = 1, \ldots, n$, *where*

$$
A := \frac{1}{4}\,(a - d + ic + ib), \quad B := \frac{1}{4}\,(a + d + ic - ib),
$$

and a, b, c, *and* d *are taken from (5.8).*

Before we come to the proof of this lemma we want to consider some interesting special situations.

The non-weighted case

In the non-weighted case $\mathbf{W}(X,Z) \equiv \mathbb{E}^{n+2}$ this identity reduces to

$$\partial_{\overline{w}}\mathscr{H}_\sigma = 2\partial_w H_\sigma W + \sum_{\vartheta=1}^{n} \left\{ (L_{\vartheta,22} + iL_{\vartheta,12})T^\sigma_{\vartheta,1} - (L_{\vartheta,21} + iL_{\vartheta,11})T^\sigma_{\vartheta,2} \right\} .$$

Thus $\mathscr{H}_\sigma$ *satisfies a system of inhomogeneous Cauchy-Riemann equations.* In particular, $\mathscr{H}_\sigma$ *is holomorphic if* X *represents a minimal surface satisfying* $H_\sigma \equiv 0$ *with flat normal bundle.*

Surfaces in $\mathbb{R}^3$

Consider an immersion $X: B \to \mathbb{R}^3$ with unit normal vector N, and as usual we write $\mathscr{H}$ instead of $\mathscr{H}_\sigma$. The lemma states

$$\partial_{\overline{w}}\mathscr{H} = A\mathscr{H} + B\overline{\mathscr{H}} + 2(A + B - \Omega_{21}^2 - i\Omega_{11}^2)H_W W + 2H_{W,w}W.$$

In particular, two cases are of special interest.

1. *Weighted minimal surface*
   Let $X: B \to \mathbb{R}^3$ be a weighted minimal surface satisfying $H_W \equiv 0$, then $\mathscr{H}$ *is a pseudo-holomorphic function* fulfilling
   $$\partial_{\overline{w}}\mathscr{H} = A\mathscr{H} + B\overline{\mathscr{H}} .$$
   In the section after the next we will see that this fact allows us to draw some important conclusions about the geometry of the spherical mapping of weighted minimal surfaces in $\mathbb{R}^3$.
2. *Surfaces with constant mean curvature*
   Let $X: B \to \mathbb{R}^3$ represent an immersion with constant scalar mean curvature H. Then $\mathscr{H}$ *is holomorphic with*
   $$\partial_{\overline{w}}\mathscr{H} = 2H_w W \equiv 0.$$
   As we will see shortly
   $$|\mathscr{H}|^2 = 4(H^2 - K),$$
   and thus either $H^2 - K \equiv 0$, equivalent to $\kappa_1 \equiv \kappa_2$, or $\kappa_1 = \kappa_2$ at most at isolated points in every compact subset $\Omega \subset\subset \mathring{B}$ (see e.g. Hopf [94], or Jost [102]). Such surface points are called *umbilical points.*

Now let us come to the proof of the previous lemma.

Proof. We consider the auxiliary function

$$\mathscr{H}_\sigma^* := L_{\sigma,11} - iL_{\sigma,12} = \frac{1}{2}\,\mathscr{H}_\sigma + WH_{W,\sigma}\,. \tag{5.12}$$

On the one hand it holds

$$\begin{aligned} A\mathscr{H}_\sigma^* + B\overline{\mathscr{H}_\sigma^*} &= \frac{1}{4}\,(a-d+ic+ib)(L_{\sigma,11} - iL_{\sigma,12}) \\ &\quad + \frac{1}{4}\,(a+d+ic-ib)(L_{\sigma,11} + iL_{\sigma,12}) \\ &= \frac{1}{2}\,(aL_{\sigma,11} + bL_{\sigma,12}) + \frac{i}{2}\,(cL_{\sigma,11} + dL_{\sigma,12}), \end{aligned}$$

while on the other hand, on account of (5.10), we infer

$$\begin{aligned} \partial_{\overline{w}}\mathscr{H}_\sigma^* &= \frac{1}{2}\,(L_{\sigma,11,u} + L_{\sigma,12,v}) + \frac{i}{2}\,(L_{\sigma,11,v} - L_{\sigma,12,u}) \\ &= \frac{1}{2}\,(aL_{\sigma,11} + bL_{\sigma,12}) + \frac{i}{2}\,(cL_{\sigma,11} + dL_{\sigma,12}) + \frac{1}{2}\,r_\sigma + \frac{i}{2}\,s_\sigma\,. \end{aligned}$$

Comparing both identities yields

$$\partial_{\overline{w}}\mathscr{H}_\sigma^* = A\mathscr{H}^* + B\overline{\mathscr{H}_\sigma^*} + \frac{1}{2}\,r_\sigma + \frac{i}{2}\,s_\sigma\,,$$

or equivalently, together with (5.12),

$$\begin{aligned} &\frac{1}{2}\,\partial_{\overline{w}}\mathscr{H}_\sigma + \partial_{\overline{w}}(WH_{W,\sigma}) \\ &\quad = \partial_{\overline{w}}\mathscr{H}_\sigma^* = \frac{A}{2}\,\mathscr{H}_\sigma + \frac{B}{2}\,\overline{\mathscr{H}_\sigma} + (A+B)WH_{W,\sigma} + \frac{1}{2}\,r_\sigma + \frac{i}{2}\,s_\sigma\,. \end{aligned}$$

Rearranging for $\partial_{\overline{w}}\mathscr{H}_\sigma$ gives

$$\partial\overline{w}\mathscr{H}_\sigma = A\mathscr{H}_\sigma + B\overline{\mathscr{H}_\sigma} + 2(A+B)WH_{W,\sigma} + r_\sigma + is_\sigma - 2\partial_{\overline{w}}(WH_{W,\sigma}).$$

Finally notice that from (5.9) if follows that

$$\begin{aligned} r_\sigma + is_\sigma - 2(WH_{g,\sigma})_{\overline{w}} &= 2W\partial_w H_{W,\sigma} - 2\Omega_{21}^2 WH_{W,\sigma} - 2i\Omega_{11}^2 WH_{W,\sigma} \\ &\quad + \sum_{\vartheta=1}^n (L_{\vartheta,22} + iL_{\vartheta,21})T_{\vartheta,1}^\sigma - \sum_{\vartheta=1}^n (L_{\vartheta,21} + iL_{\vartheta,11})T_{\vartheta,2}^\sigma\,, \end{aligned}$$

and the statement is proved. □

5.7 A Pascali system for the Hopf functions

Let $X\colon B \to \mathbb{R}^{n+2}$ be a weighted minimal surface satisfying $H_W \equiv 0$ with given weight matrix $\mathbf{W}(X,Z)$. We want to rearrange the above compex-valued differential system into a so-called *Pascali system.*

Definition 5.5. We define the *complex-valued torsion vector*

$$T_\sigma^\vartheta := T_{\sigma,1}^\vartheta + iT_{\sigma,2}^\vartheta \in \mathbb{C} \quad \text{for } \sigma,\vartheta = 1,\ldots,n. \tag{5.13}$$

Now using weighted conformally parameters $(u,v) \in B$ we calculate

$$\begin{aligned}
&(L_{\vartheta,22} + iL_{\vartheta,12})T_{\vartheta,1}^\sigma - (L_{\vartheta,12} + iL_{\vartheta,11})T_{\vartheta,2}^\sigma \\
&\quad = 2H_{W,\vartheta}WT_{\vartheta,1}^\sigma - (L_{\vartheta,11} - iL_{\vartheta,12})T_{\vartheta,1}^\sigma - i(L_{\vartheta,11} - iL_{\vartheta,12})T_{\vartheta,2}^\sigma \\
&\quad = 2H_{W,\vartheta}WT_{\vartheta,1}^\sigma - \mathscr{H}_\vartheta^*(T_{\vartheta,1}^\sigma + iT_{\vartheta,2}^\sigma) \\
&\quad = 2H_{W,\vartheta}WT_{\vartheta,1}^\sigma - \frac{1}{2}\mathscr{H}_\vartheta T_\vartheta^\sigma - H_{W,\vartheta}WT_\vartheta^\sigma
\end{aligned}$$

recalling the identity

$$\mathscr{H}_\vartheta^* = L_{\vartheta,11} - iL_{\vartheta,12} = \frac{1}{2}\mathscr{H}_\vartheta + H_{W,\vartheta}W$$

from the previous section. Now $H_W \equiv 0$ gives us

$$(L_{\vartheta,22} + iL_{\vartheta,12})T_{\vartheta,1}^\sigma - (L_{\vartheta,12} + iL_{\vartheta,11})T_{\vartheta,2}^\sigma = -\frac{1}{2}\mathscr{H}_\vartheta T_\vartheta^\sigma .$$

Finally we need a suitable complex-valued Hopf vector. Our central definition of this section is

Definition 5.6. The *Hopf vector* $\mathscr{H} \in \mathbb{C}^n$ of an immersion $X\colon B \to \mathbb{R}^{n+2}$ w.r.t. some ONF N is defined as

$$\mathscr{H} := (\mathscr{H}_1,\ldots,\mathscr{H}_n) \in \mathbb{C}^n . \tag{5.14}$$

Now we come to the main result of this section.

Theorem 5.4. *Let the weighted conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then it holds*

$$\begin{aligned}
\partial_{\overline{w}}\mathscr{H}_\sigma &= A\mathscr{H}_\sigma + B\overline{\mathscr{H}}_\sigma - \frac{1}{2}\sum_{\vartheta=1}^n T_\vartheta^\sigma \mathscr{H}_\vartheta \\
&= A\mathscr{H}_\sigma + B\overline{\mathscr{H}}_\sigma + \frac{1}{2}\sum_{\vartheta=1}^n T_\sigma^\vartheta \mathscr{H}_\vartheta
\end{aligned} \tag{5.15}$$

for $\sigma = 1,\ldots,n$.

The Hopf vector $\mathscr{H} \in \mathbb{C}^n$ satisfies the Pascali system

$$\partial_{\overline{w}}\mathscr{H} = A\mathscr{H} + B\overline{\mathscr{H}} + \frac{1}{2}\mathbf{T}\circ\mathscr{H} \tag{5.16}$$

with the complex-valued torsion matrix

$$\mathbf{T} := (T_\sigma^\vartheta)_{\sigma,\vartheta=1,\ldots,n} \in \mathbb{C}^{n\times n}\,.$$

Here we take into account

$$T_\vartheta^\sigma = -T_\sigma^\vartheta\,.$$

But now notice that

$$\begin{aligned}|\mathscr{H}_\sigma|^2 &= (L_{\sigma,11}-L_{\sigma,22})^2 + 4L_{\sigma,12}^2 = (L_{\sigma,11}+L_{\sigma,22})^2 + 4(L_{\sigma,11}L_{\sigma,22}-L_{\sigma,12}^2)\\ &= 4(H_{W,\sigma}^2 - K_\sigma)W^2 = -4K_\sigma W^2\,.\end{aligned}$$

Thus we arrive at the interesting identity

$$|\mathscr{H}|^2 = \sum_{\sigma=1}^n |\mathscr{H}_\sigma|^2 = -4\sum_{\sigma=1}^n K_\sigma W^2 = -4KW^2\,,$$

i.e. *the zeros of the Gauss curvature K of a weighted minimal surface are the zeros of the complex-valued Hopf vector $\mathscr{H}$*!

By the way, we also infer

$$|\mathscr{H}|^2 = 4\sum_{\sigma=1}^n (H_\sigma^2 - K_\sigma)W^2 = 4(H^2-K)W^2\,,$$

and identity used already in the previous section.

Actually the complex-valued system (5.16) is a special *Pascali system* for our Hopf function $\mathscr{H}$. Following Wendland [164], Theorem 5.3.3, $\mathscr{H}$ can locally be represented in the form

$$\mathbf{E}\circ\boldsymbol{\Phi}$$

with a C^α-regular matrix $\mathbf{E}\in\mathbb{C}^{n\times n}$ with the property $\det\mathbf{E}\neq 0$, and a holomorphic vector $\boldsymbol{\Phi}\in\mathbb{C}^n$.

Consequently we can apply the *similarity principle for generalized analytic vector-valued functions* (see e.g. Wendland [164], Theorem 5.3.5) and arrive at the following characterization of weighted minimal surfaces.

Corollary 5.2. *Either it holds $\mathscr{H}\equiv 0$ in B, or $\mathscr{H}$ has only isolated zeros of finite order in every compact subset $\Omega\subset\subset\mathring{B}$. Therefore, either it holds $K\equiv 0$ in B and the weighted minimal surface is a plane, or in every such $\Omega\subset\subset\mathring{B}$ there are only finitely many points with $K=0$.*

5.8 An example: Weighted minimal surfaces in $\mathbb{R}^3$

Consider a weighted minimal surface $X\colon B \to \mathbb{R}^3$ with weight $\mathbf{W}(X,Z) \in \mathbb{R}^{3\times 3}$ and unit normal vector

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}\,.$$

The associated Hopf function $\mathscr{H}$ satisfies

$$\mathscr{H}_{\overline{w}} = A\mathscr{H} + B\overline{\mathscr{H}}\,.$$

Solutions of a such differential equation are called *pseudoholomorphic functions.*

Following the *similarity principle* for pseudoholomorphic functions of Bers and Vekua (see e.g. Bers [14], Courant and Hilbert [40], Sauvigny [143], Vekua [161], or Wendland [164]), the Hopf function $\mathscr{H}$ can be represented in the form

$$\Phi(w)e^{\Psi(w)}$$

with a holomorphic function Φ and some integral function Ψ. *Again the zeros of the Gaussian curvature K are isolated in every compact subset $\Omega \subset\subset \mathring{B}$ as long as $X\colon B \to \mathbb{R}^3$ is not a plane.*

In the following we want to show that the zeros of K coincide to the *branch points* of the spherical mapping N which are characterized by the property

$$N_u \times N_v = 0 \quad \text{at points with } K = 0.$$

This tells us the following calculation: Introduce weighted conformal parameters $(u,v) \in B$. Then the Weingarten equations in the weighted form from section 4.3 can be written as

$$\begin{aligned} N_u &= -\frac{L_{11}}{W}\,\mathbf{W}(X,N)\circ X_u - \frac{L_{12}}{W}\,\mathbf{W}(X,N)\circ X_v\,,\\ N_v &= -\frac{L_{12}}{W}\,\mathbf{W}(X,N)\circ X_u - \frac{L_{22}}{W}\,\mathbf{W}(X,N)\circ X_v\,. \end{aligned}$$

We need the following calculus rule from Sauvigny [141].

Lemma 5.3. *Let $\mathbf{M} \subset \mathbb{R}^{3\times 3}$ be a non-singular and symmetric matrix. Then*

$$\big(\mathbf{M}\circ X\big) \times \big(\mathbf{M}\circ Y\big) = \big(\det\mathbf{M}\big)\mathbf{M}^{-1}\circ (X \times Y)$$

for arbitrary $X,Y \in \mathbb{R}^3$.

Proof. For an arbitrary vector $Z \in \mathbb{R}^3$ we calculate

$$\begin{aligned}
&\big\{(\mathbf{M}\circ X)\times(\mathbf{M}\circ Y)-(\det\mathbf{M})\mathbf{M}^{-1}\circ(X\times Y)\big\}\cdot(\mathbf{M}\circ Z)\\
&\qquad=\big\{(\mathbf{M}\circ X)\times(\mathbf{M}\circ Y)\big\}\cdot(\mathbf{M}\circ Z)-(\det\mathbf{M})(X\times Y)\cdot(\mathbf{M}^{-1}\circ\mathbf{M}\circ Z)\\
&\qquad=(\det\mathbf{M})\big\{(X\times Y)\cdot Z-(X\times Y)\cdot Z\big\}=0.
\end{aligned}$$

This proves the calculus rule. □

At last we obtain

$$\begin{aligned}
N_u\times N_v &= \left(\frac{L_{11}}{W}\,\mathbf{W}(X,N)\circ X_u+\frac{L_{12}}{W}\,\mathbf{W}(X,N)\circ X_v\right)\times\ldots\\
&\qquad\ldots\times\left(\frac{L_{12}}{W}\,\mathbf{W}(X,N)\circ X_u+\frac{L_{22}}{W}\,\mathbf{W}(X,N)\circ X_v\right)\\
&= \frac{L_{11}L_{22}-L_{12}^2}{W^2}\,\big\{\mathbf{W}(X,Z)\circ X_u\big\}\times\big\{\mathbf{W}(X,Z)\circ X_v\big\}\\
&= \frac{L_{11}L_{22}-L_{12}^2}{W^2}\,(X_u\times X_v)=KWN
\end{aligned}$$

making use of $\mathbf{W}(X,N)\circ N=N$. Thus, as stated, surface points with the property $N_u\times N_v=0$ are exactly points with $K=0$.

In chapter 15 we will investigate more closely weighted minimal surfaces as well as critical points for general anisotropic and inhomogeneous functionals in $\mathbb{R}^3$.

Chapter 6
The Gauss-Osserman map

This chapter is devoted to Grassmann forms and the Gauss-Osserman mapping as a further method to generalize the classical spherical mapping of surfaces. This particularly proves its strength in minimal surface theory. The natural metrical background for these studies is the Fubini-Study metric. Finally we discuss a new curvature vector of the normal bundle.

6.1 The exterior product

For the following algebraic concepts of the Grassmann geometry we refer to Cartan [23] or Heil [78]; see also Grassmann [74].

Definition 6.1. Let $n \ge 1$. The *exterior product*

$$\wedge\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^N, \quad N = \binom{n}{2} = \frac{n(n-1)}{2},$$

is defined by means of the following rules:

(E1) The mapping $\mathbb{R}^n \ni (v,w) \mapsto v \wedge w \in \mathbb{R}^N$ is bilinear, i.e.

$$\begin{aligned}&(\alpha_1 v_1 + \alpha_2 v_2) \wedge (\beta_1 w_1 + \beta_2 w_2)\\ &\quad = \alpha_1\beta_2\, v_1 \wedge w_1 + \alpha_1\beta_2\, v_1 \wedge w_2 + \alpha_2\beta_1 v_2 \wedge w_1 + \alpha_2\beta_2 v_2 \wedge w_2\end{aligned}$$

for all $\alpha_i, \beta_i \in \mathbb{R}$, $v_i, w_i \in \mathbb{R}^n$, and it is skew-symmetric,

$$v \wedge w = -w \wedge v \quad \text{for all } v, w \in \mathbb{R}^n;$$

in particular, it holds $v \wedge v = 0$.

(E2) With $e_1 = (1,0,0,\ldots,0) \in \mathbb{R}^n$, $e_2 = (0,1,0,\ldots,0) \in \mathbb{R}^n$ etc. we set

$$\begin{aligned} e_1 \wedge e_2 &:= (1,0,0,\ldots,0,0) \in \mathbb{R}^N,\\ e_1 \wedge e_3 &:= (0,1,0,\ldots,0,0) \in \mathbb{R}^N,\\ &\;\vdots\\ e_{n-1} \wedge e_n &:= (0,0,0,\ldots,0,1) \in \mathbb{R}^N. \end{aligned}$$

From this settings we immediately obtain the

Proposition 6.1. *The vectors $e_k \wedge e_\ell$ form an orthonormal basis of $\mathbb{R}^N$, i.e.*

$$(e_i \wedge e_j) \cdot (e_k \wedge e_\ell) = \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{if } i \neq k \text{ or } j \neq \ell \end{cases} . \tag{6.1}$$

Proposition 6.2. *For two vectors $v = (v^1, \ldots, v^n)$ and $w = (w^1, \ldots, w^n)$ it holds*

$$v \wedge w = \sum_{1 \leq i < j \leq n} (v^i w^j - v^j w^i) e_i \wedge e_j \,. \tag{6.2}$$

Proof. We compute

$$\begin{aligned} v \wedge w &= \left(\sum_{i=1}^n v^i e_i \right) \wedge \left(\sum_{j=1}^n w^j e_j \right) = \sum_{i,j=1}^n v^i w^j e_i \wedge e_j \\ &= \sum_{1 \leq i < j \leq n} (v^i w^j - v^j w^i) e_i \wedge e_j \,, \end{aligned}$$

which already proves the statement. □

Example 6.1. For $n = 3$ we have $N = 3$ and

$$e_1 \wedge e_2 = (1,0,0), \quad e_1 \wedge e_3 = (0,1,0), \quad e_2 \wedge e_3 = (0,0,1).$$

With two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ we compute

$$\begin{aligned} v \wedge w &= v_1 w_2 \, e_1 \wedge e_2 - v_1 w_3 \, e_1 \wedge e_3 + v_2 w_1 \, e_2 \wedge e_1 + v_2 w_3 \, e_2 \wedge e_3 \\ &\quad - v_3 w_1 \, e_3 \wedge e_1 + v_3 w_2 \, e_3 \wedge e_2 \\ &= (v_1 w_2 - v_2 w_1) e_1 \wedge e_2 + (v_3 w_1 - v_1 w_3) e_1 \wedge e_3 + (v_2 w_3 - v_3 w_2) e_2 \wedge e_3 \\ &= (v_1 w_2 - v_2 w_1, v_3 w_1 - v_1 w_3, v_2 w_3 - v_3 w_2). \end{aligned}$$

Thus *the usual vector product*

$$v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

in $\mathbb{R}^3$ does not coincide with the exterior product $v \wedge w$.

We want to collect some further properties of the exterior product.

Corollary 6.1. *For arbitrary vectors $a, b \in \mathbb{R}^n$ there hold*

- $(\lambda a) \wedge b = \lambda (a \wedge b)$;
- $(a + b) \wedge c = a \wedge c + b \wedge c$;
- $(a \wedge b)_{u^i} = a_{u^i} \wedge b + a \wedge b_{u^i}$.

Let two vectors $v=(v_1,v_2,0,\ldots,0)$ *and* $w=(w_1,w_2,0,\ldots,0)$ *be given. Then*

- $v\wedge w\perp \operatorname{span}\big\{v\wedge e_3,\ldots,v\wedge e_n,w\wedge e_3,\ldots,w\wedge e_n,e_3\wedge e_n,\ldots,e_{n-1}\wedge e_n\big\}.$

6.2 The Grassmann normal space. Grassmann forms

Let $X\colon B\to\mathbb{R}^{n+2}$ be conformally parametrized, N an associated ONF. Then the set

$$\left\{\frac{X_u}{\sqrt{W}},\frac{X_v}{\sqrt{W}},N_1,N_2,\ldots,N_n\right\}$$

forms an orthonormal system spanning the embedding space $\mathbb{R}^{n+2}$ at each point of the surface X.

Definition 6.2. The *Grassmann normal space* of the conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$ with ONF N at the point $w\in B$ is given by

$$\mathfrak{G}_w[X]:=\operatorname{Span}\big\{\mathscr{N},\mathscr{X}_{11},\ldots,\mathscr{X}_{1n},\mathscr{X}_{21},\ldots,\mathscr{X}_{2n},\mathscr{N}_{11},\ldots,\mathscr{N}_{1n},\mathscr{N}_{23},\ldots,\mathscr{N}_{n-1,n}\big\}$$

with the $N=\binom{n}{2}$ unit vectors

$$\mathscr{N}:=\frac{X_u\wedge X_v}{W}\,,\quad \mathscr{X}_{i\sigma}:=\frac{X_{u^i}\wedge N_\sigma}{\sqrt{W}}\,,\quad \mathscr{N}_{\sigma\vartheta}:=N_\sigma\wedge N_\vartheta \tag{6.3}$$

for $i=1,2$ and $\sigma,\vartheta=1,\ldots,n$.

Let us consider some examples.

1. In case $n=1$ of one codimension we have simply

$$\mathscr{N}=N,\quad \mathscr{X}_{11}=\frac{X_v}{\sqrt{W}}\,,\quad \mathscr{X}_{21}=-\frac{X_u}{\sqrt{W}}\,.$$

   Thus the Grassmann space is just $\operatorname{Span}\{X_u,X_v,N\}\cong\mathbb{R}^3$.

2. For further illustration we consider the case $n=2$ of two codimensions. The Grassmann space $\mathfrak{G}_w[X]$ consist of

$$\mathscr{N}=\frac{X_u\wedge X_v}{W}\,,$$

$$\mathfrak{X}_{11}=\frac{X_u\wedge N_1}{\sqrt{W}}\,,\quad \mathfrak{X}_{12}=\frac{X_u\wedge N_2}{\sqrt{W}}\,,\quad \mathfrak{X}_{21}=\frac{X_v\wedge N_1}{\sqrt{W}}\,,\quad \mathfrak{X}_{22}=\frac{X_v\wedge N_2}{\sqrt{W}}\,,$$

$$\mathscr{N}_{12}=N_1\wedge N_2\,.$$

3. Let a weight matrix $\mathbf{W}(X,Z)$ be given. We introduce weighted conformal parameters. The system
$$\left\{\frac{1}{\sqrt{W}}X_u,\frac{1}{\sqrt{W}}X_v,N_1,\ldots,N_n\right\}$$
must then be replaced by the following orthonormal moving frame
$$\left\{\frac{1}{\sqrt{W}}\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u,\frac{1}{\sqrt{W}}\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v,N_1,\ldots,N_n\right\}$$
where $\mathbf{W}(X,Z)^{\frac{1}{2}}$ is defined via the spectral decomposition of $\mathbf{W}(X,Z)$.

From $|\mathscr{N}|^2=1$ we immediately obtain $\mathscr{N}_{u^i}\perp\mathscr{N}$ for $i=1,2$. We want to prove a representation formula for the derivatives $\mathscr{N}_{u^i}$ similar to the Weingarten equations.

Theorem 6.1. *(Grassmann-Weingarten equations)*
Let the conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$ together with an ONF N be given. Then there hold
$$\mathscr{N}_{u^i}=-\sum_{m=1}^{2}\sum_{\vartheta=1}^{n}\mathscr{L}_i^{m\vartheta}\,\mathscr{X}_{m\vartheta} \tag{6.4}$$
for $i=1,2$ with the 2nd (Grassmann-type) $\mathscr{G}$-fundamental form
$$\mathscr{L}_i^{m\vartheta}:=-\mathscr{N}_{u^i}\cdot\mathscr{X}_{m\vartheta}=\frac{1}{\sqrt{W}}\left(\frac{L_{\vartheta,i1}}{W}X_u\wedge X_{u^m}+\frac{L_{\vartheta,i2}}{W}X_v\wedge X_{u^m}\right)\cdot\mathscr{N}.$$
In particular, there hold
$$\begin{aligned}\mathscr{L}_1^{1\vartheta}&=-\frac{L_{\vartheta,12}}{\sqrt{W}}, \quad \mathscr{L}_1^{2\vartheta}=\frac{L_{\vartheta,11}}{\sqrt{W}},\\ \mathscr{L}_2^{1\vartheta}&=-\frac{L_{\vartheta,22}}{\sqrt{W}}, \quad \mathscr{L}_2^{2\vartheta}=\frac{L_{\vartheta,12}}{\sqrt{W}}.\end{aligned} \tag{6.5}$$

Proof. We already know $\mathscr{N}_{u^i}\cdot\mathscr{N}=0$ for $i=1,2$. Thus we make the ansatz
$$\mathscr{N}_{u^i}=\sum_{m=1}^{2}\sum_{\vartheta=1}^{n}a_i^{m\vartheta}\,\mathscr{X}_{m\vartheta}+\sum_{\sigma,\vartheta=1}^{n}b_i^{\sigma\vartheta}\,\mathscr{N}_{\sigma\vartheta}.$$
From $\mathscr{N}\cdot\mathscr{N}_{\sigma\vartheta}=0$ we infer
$$\begin{aligned}b_i^{\sigma\vartheta}&=\mathscr{N}_{u^i}\cdot\mathscr{N}_{\sigma\vartheta}=-\mathscr{N}\cdot\mathscr{N}_{\sigma\vartheta,u^i}=-\mathscr{N}\cdot(N_\sigma\wedge N_\vartheta)_{u^i}\\ &=-\mathscr{N}\cdot(N_{\sigma,u^i}\wedge N_\vartheta+N_\sigma\wedge N_{\vartheta,u^i}).\end{aligned}$$

But the Weingarten equations (2.11) imply

$$N_{\sigma,u^i} \wedge N_\vartheta = -\frac{L_{\sigma,i1}}{W} X_u \wedge N_\vartheta - \frac{L_{\sigma,i2}}{W} X_v \wedge N_\vartheta + \sum_{\omega=1}^{n} T_{\sigma,i}^{\omega} N_\omega \wedge N_\vartheta$$
$$= -\frac{L_{\sigma,i1}}{W} \mathscr{X}_{1\vartheta} - \frac{L_{\sigma,i2}}{W} \mathscr{X}_{2\vartheta} + \sum_{\omega=1}^{n} T_{\sigma,i}^{\omega} \mathscr{N}_{\omega\vartheta}$$

such that $N_{\sigma,u^i} \wedge N_\vartheta \perp \mathscr{N}$, and $b_i^{\sigma\vartheta} \equiv 0$. We now determine the coefficients $a_i^{m\vartheta}$:

$$a_i^{m\vartheta} = \mathscr{N}_{u^i} \cdot \mathscr{X}_{m\vartheta} = -\mathscr{N} \cdot \mathscr{X}_{m\vartheta,u^i}$$
$$= -\frac{1}{\sqrt{W}} \mathscr{N} \cdot (X_{u^i u^m} \wedge N_\vartheta + X_{u^m} \wedge N_{\vartheta,u^i}) - \left(\frac{\partial}{\partial u^i} \frac{1}{\sqrt{W}}\right) \mathscr{N} \cdot (X_{u^m} \wedge N_\vartheta)$$
$$= -\frac{1}{\sqrt{W}} \mathscr{N} \cdot (X_{u^m} \wedge N_{\vartheta,u^i})$$
$$= \frac{1}{\sqrt{W}} \mathscr{N} \cdot \left(\frac{L_{\vartheta,i1}}{W} X_{u^m} \wedge X_u + \frac{L_{\vartheta,i2}}{W} X_{u^m} \wedge X_v\right).$$

Then we set $\mathscr{L}_i^{m\vartheta} := -a_i^{m\vartheta}$. Furthermore, we examplarily compute

$$\mathscr{L}_1^{1\vartheta} = a_1^{1\vartheta} = \frac{1}{\sqrt{W}} \frac{X_u \wedge X_v}{W} \frac{L_{\vartheta,12}}{W} X_u \wedge X_v = \frac{L_{\vartheta,12}}{\sqrt{W}}$$

and analogously for the other coefficients. This proves the theorem. $\square$

We want to point out the similarity to the classical Weingarten equations

$$N_{u^i} = -\frac{L_{11}}{W} X_u - \frac{L_{22}}{W} X_v$$

for a conformally parametrized immersion $X\colon B \to \mathbb{R}^3$ with unit normal vector N. This motivates to consider $\mathscr{N}$ as a possible generalisation of N if the codimension of the surface is greater the 1.

6.3 Curvature vector and curvature matrix of the normal bundle

We want to use this Grassmann formalism to reformulate Definition 3.5 using the exterior product. We also take advantage of the opportunity to introduce further useful quantities describing the curvature of the normal bundle.

Our aim is to prove the

Proposition 6.3. *Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ be given. Then the Grassmann curvature vector $\mathscr{S}$ of its normal bundle*

$$\mathscr{S}:=\frac{1}{W}\sum_{1\le\sigma<\vartheta\le n}S_{\sigma,12}^{\vartheta}N_\sigma\wedge N_\vartheta$$

does neither depend on the parametrization nor on the choice of the normal frame. In particular, $\mathscr{S}$ and its length

$$|\mathscr{S}|=\sqrt{\mathscr{S}\cdot\mathscr{S}}=\sqrt{S\cdot S}=\frac{1}{W}\sqrt{\sum_{1\le\sigma<\vartheta\le n}(S_{\sigma,12}^{\vartheta})^2}$$

represent a geometric quantities.

Here we recall the definition

$$S=\frac{1}{W}\left(S_{1,12}^2,S_{1,12}^3,\ldots,S_{n-1,12}^n\right)\in\mathbb{R}^N$$

of the curvature vector of the normal bundle.

The curvature vector $\mathscr{S}$ (or just S) is a third important curvature quantity for twodimensional surfaces in $\mathbb{R}^{n+2}$ beside their Gaussian curvature K and their mean curvature vector H :

$$\begin{aligned}
K &= \sum_{\sigma=1}^{n}\frac{L_{\sigma,11}L_{\sigma,22}-L_{\sigma,12}^2}{g_{11}g_{22}-g_{12}^2}\in\mathbb{R}\,,\\
H &= \sum_{\sigma=1}^{n}\frac{L_{\sigma,11}g_{22}-2L_{\sigma,12}g_{12}+L_{\sigma,22}g_{11}}{g_{11}g_{22}-g_{12}^2}N_\sigma\in\mathbb{R}^n\,,\\
\mathscr{S} &= \frac{1}{W}\sum_{1\le\sigma<\vartheta\le n}S_{\sigma,12}^{\vartheta}N_\sigma\wedge N_\vartheta\in\mathbb{R}^N\,.
\end{aligned}$$

In case $n=2$ we have

$$\mathscr{S}=\frac{1}{W}S_{1,12}^2N_1\wedge N_2\quad\text{or}\quad S=\frac{1}{W}S_{1,12}^2\,.$$

To prove at last the above proposition we are only left with verifying the length-invariance of $\mathscr{S}$. For this purpose we make the following

Definition 6.3. Let the immersion $X\colon B\to\mathbb{R}^{n+2}$ together with an ONF N be given. Then the *torsion matrices of* N are defined as

$$\mathbf{T}_i=(T_{\sigma,i}^{\vartheta})_{\sigma,\vartheta=1,\ldots,n}\in\mathbb{R}^{n\times n}\quad\text{for } i=1,2.$$

Furthermore, the *curvature matrix of its normal bundle* is given as

$$\mathbf{S}_{12}=(S_{\sigma,12}^{\vartheta})_{\sigma,\vartheta=1,\ldots,n}\in\mathbb{R}^{n\times n}\,.$$

Recall the setting $\mathbf{R}=(r_{\sigma\vartheta})_{\sigma,\vartheta=1,\ldots,n}\in\mathbb{R}^{n\times n}$ for the orthogonal transformation between ONFs. Let $\mathbf{R}^T$ denote its transposition such that $\mathbf{R}\circ\mathbf{R}^T=\mathbf{R}^T\circ\mathbf{R}=\mathbb{E}^n$, and let $\widetilde{\mathbf{S}}_{12}$ be the curvature matrix after applying $\mathbb{R}$ to an ONF N.

Lemma 6.1. *It holds the transformation rule*

$$\widetilde{\mathbf{S}}_{12}=\mathbf{R}\circ\mathbf{S}_{12}\circ\mathbf{R}^T\,.$$

Proof. For the proof we consider

$$\begin{aligned}\widetilde{T}_{\sigma,i}^{\vartheta}=\widetilde{N}_{\sigma,u^i}\cdot\widetilde{N}_{\vartheta}&=\sum_{\alpha=1}^{n}\big(r_{\sigma\alpha,u^i}N_\alpha+r_{\sigma\alpha}N_{\alpha,u^i}\big)\cdot\sum_{\beta=1}^{n}r_{\vartheta\beta}N_\beta\\&=\sum_{\alpha,\beta=1}^{n}\big(r_{\sigma\alpha,u^i}r_{\vartheta\beta}\delta_{\alpha\beta}+r_{\sigma\alpha}r_{\vartheta\beta}T_{\alpha,i}^{\beta}\big)\\&=\sum_{\alpha=1}^{n}r_{\sigma\alpha,u^i}r^{\alpha\vartheta}+\sum_{\alpha,\beta=1}^{n}r_{\sigma\alpha}T_{\alpha,i}^{\beta}r^{\beta\vartheta}\,.\end{aligned}$$

Thus we arrive at the rule

$$\widetilde{\mathbf{T}}_i=\mathbf{R}_{u^i}\circ\mathbf{R}^T+\mathbf{R}\circ\mathbf{T}_i\circ\mathbf{R}^T\,.$$

Using this formula we evaluate

$$\widetilde{\mathbf{S}}_{12}=\widetilde{\mathbf{T}}_{1,v}-\widetilde{\mathbf{T}}_{2,u}-\widetilde{\mathbf{T}}_1\circ\widetilde{\mathbf{T}}_2^T+\widetilde{\mathbf{T}}_2\circ\widetilde{\mathbf{T}}_1^T\,.$$

Namely, first we have

$$\begin{aligned}\widetilde{\mathbf{T}}_{1,v}-\widetilde{\mathbf{T}}_{2,u}&=(\mathbf{R}_u\circ\mathbf{R}^T+\mathbf{R}\circ\mathbf{T}_1\circ\mathbf{R}^T)_v-(\mathbf{R}_v\circ\mathbf{R}^T+\mathbf{R}\circ\mathbf{T}_2\circ\mathbf{R}^T)_u\\&=\mathbf{R}_u\circ\mathbf{R}_v^T-\mathbf{R}_v\circ\mathbf{R}_u^T+\mathbf{R}\circ(\mathbf{T}_{1,v}-\mathbf{T}_{2,u})\circ\mathbf{R}^T+\mathbf{R}_v\circ\mathbf{T}_1\circ\mathbf{R}^T\\&\quad+\mathbf{R}\circ\mathbf{T}_1\circ\mathbf{R}_v^T-\mathbf{R}_u\circ\mathbf{T}_2\circ\mathbf{R}^T-\mathbf{R}\circ\mathbf{T}_2\circ\mathbf{R}_u^T\,,\end{aligned}$$

and furthermore

$$\begin{aligned}&\widetilde{\mathbf{T}}_1\circ\widetilde{\mathbf{T}}_2^T-\widetilde{\mathbf{T}}_2\circ\widetilde{\mathbf{T}}_1^T\\&\quad=(\mathbf{R}_u\circ\mathbf{R}^T+\mathbf{R}\circ\mathbf{T}_1\circ\mathbf{R}^T)\circ(\mathbf{R}\circ\mathbf{R}_v^T+\mathbf{R}\circ\mathbf{T}_2^T\circ\mathbf{R}^T)\\&\qquad-(\mathbf{R}_v\circ\mathbf{R}^T+\mathbf{R}\circ\mathbf{T}_2\circ\mathbf{R}^T)\circ(\mathbf{R}\circ\mathbf{R}_u^T+\mathbf{R}\circ\mathbf{T}_1^T\circ\mathbf{R}^T)\\&\quad=\mathbf{R}_u\circ\mathbf{R}_v^T+\mathbf{R}_u\circ\mathbf{T}_2^T\circ\mathbf{R}^T+\mathbf{R}\circ\mathbf{T}_1\circ\mathbf{R}_v^T+\mathbf{R}\circ\mathbf{T}_1\circ\mathbf{T}_2^T\circ\mathbf{R}^T\\&\qquad-\mathbf{R}_v\circ\mathbf{R}_u^T-\mathbf{R}_v\circ\mathbf{T}_1^T\circ\mathbf{R}^T-\mathbf{R}\circ\mathbf{T}_2\circ\mathbf{R}_u^T-\mathbf{R}\circ\mathbf{T}_2\circ\mathbf{T}_1^T\circ\mathbf{R}^T\end{aligned}$$

since $\mathbf{R}\circ\mathbf{R}^T=\mathbf{R}^T\circ\mathbf{R}=\mathbb{E}^n$.

Taking both identities together gives

$$
\begin{aligned}
&\widetilde{\mathbf{T}}_{1,v} - \widetilde{\mathbf{T}}_{2,u} - \widetilde{\mathbf{T}}_1 \circ \widetilde{\mathbf{T}}_2^T + \widetilde{\mathbf{T}}_2 \circ \widetilde{\mathbf{T}}_1^T \\
&\quad = \mathbf{R} \circ (\mathbf{T}_{1,v} - \mathbf{T}_{2,u} - \mathbf{T}_1 \circ \mathbf{T}_2^T + \mathbf{T}_2 \circ \mathbf{T}_1^T) \circ \mathbf{R}^T \\
&\qquad + \mathbf{R}_v \circ \mathbf{T}_1 \circ \mathbf{R}^T + \mathbf{R} \circ \mathbf{T}_1 \circ \mathbf{R}_v^T - \mathbf{R}_u \circ \mathbf{T}_2 \circ \mathbf{R}^T - \mathbf{R} \circ \mathbf{T}_2 \circ \mathbf{R}_u^T \\
&\qquad - \mathbf{R}_u \circ \mathbf{T}_2^T \circ \mathbf{R}^T - \mathbf{R} \circ \mathbf{T}_1 \circ \mathbf{R}_v^T + \mathbf{R}_v \circ \mathbf{T}_1^T \circ \mathbf{R}^T + \mathbf{R} \circ \mathbf{T}_2 \circ \mathbf{R}_u^T \\
&\quad = \mathbf{R} \circ (\mathbf{T}_{1,v} - \mathbf{T}_{2,u} - \mathbf{T}_1 \circ \mathbf{T}_2^T + \mathbf{T}_2 \circ \mathbf{T}_1^T) \circ \mathbf{R}^T
\end{aligned}
$$

taking $\mathbf{T}_i = -\mathbf{T}_i^T$ into account. This proves the statement. □

Now we continue with the proof of the previous proposition.

Proof of the proposition. In terms of our usual $SO(n)$-action we compute

$$
\begin{aligned}
\sum_{\sigma,\vartheta=1}^{n} \widetilde{S}_{\sigma,12}^{\vartheta} \widetilde{N}_\sigma \wedge \widetilde{N}_\vartheta &= \sum_{\sigma,\vartheta=1}^{n} \sum_{\alpha,\beta=1}^{n} \widetilde{S}_{\sigma,12}^{\vartheta} r_{\sigma\alpha} r_{\vartheta\beta} N_\alpha \wedge N_\beta \\
&= \sum_{\sigma,\vartheta=1}^{n} \sum_{\alpha,\beta=1}^{n} r^{\alpha\sigma} \widetilde{S}_{\sigma,12}^{\vartheta} r_{\vartheta\beta} N_\alpha \wedge N_\beta \\
&= \sum_{\alpha,\beta=1}^{n} S_{\alpha,12}^{\beta} N_\alpha \wedge N_\beta
\end{aligned}
$$

with the orthogonal mapping $\mathbf{R} = (r_{\sigma\omega})_{\sigma,\omega=1,\ldots,n}$. This proves the proposition. □

6.4 Grassmann manifolds and Gauss-Osserman map

We follow Hoffman and Osserman [93], see also Osserman [130].

Definition 6.4. The k-dimensional *Grassmann manifold* $\mathrm{Gr}_k(m)$ is the space of all k-dimensional subspaces of $\mathbb{R}^m$.

For example, the Grassmann manifold $\mathrm{Gr}_1(2)$ is the set of all lines through the origin in $\mathbb{R}^2$. Or the Grassmann manifold $\mathrm{Gr}_2(3)$ is the set of all planes through the origin in $\mathbb{R}^3$. Such a plane is uniquely determined by the line intersecting it perpendicularly. Hence $\mathrm{Gr}_2(3)$ is isomorphic to $\mathrm{Gr}_1(3)$.

Consider now an *oriented* plane $\mathscr{P} \subset \mathbb{R}^{n+2}$ spanned by an ordered pair of orthogonal and equally long vectors $V \in \mathbb{R}^{n+2}$ and $W \in \mathbb{R}^{n+2}$. With $\mathscr{P}$ we associate the complex-valued point

$$
Z = V + iW \in \mathbb{C}^{n+2}, \quad Z = (z^1, \ldots, z^{n+2}).
$$

If $\{V', W'\}$ denotes a second basis of $\mathscr{P}$, then $Z' = cZ$ with some complex number c. This means that *each oriented plane corresponds to a uniquely determined point of the projective space* $\mathbb{C}P^{n+1}$.

We compute the complex square of Z

$$Z^2 = \sum_{k=1}^{n+2} (z^k)^2 = |V|^2 - |W|^2 + 2iV \cdot W^t = 0.$$

Thus Z is an *isotropic vector,* and it belongs to the *complex-valued quadric*

$$Q^{n+1} := \left\{ Z \in \mathbb{C}P^{n+1} \, : \, Z^2 = 0 \right\} \subset \mathbb{C}P^{n+1}.$$

Each oriented plane corresponds one-to-one to a point of the quadric Q^{n+1}.

To be more precise: Consider a conformally parametrized surface $X\colon B \to \mathbb{R}^{n+2}$, then we would set

$$V := X_u\,, \quad W := X_v\,.$$

In the Finsler-type case for surfaces equipped with a prescribed weight matrix $\mathbf{W}(X,Z)$ on the other hand we let

$$V := \mathbf{W}(X,N)^{\frac{1}{2}} \circ X_u \quad \text{and} \quad W := \mathbf{W}(X,N)^{\frac{1}{2}} \circ X_v.$$

This leads us to our next

Definition 6.5. Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ be given. Then its *Gauss-Osserman map* is defined as the complex-valued mapping

$$\mathfrak{N}\colon B \longrightarrow Q^{n+1}\,, \quad \mathfrak{N}(w) := X_u(w) + iX_v(w)\,, \quad w \in B. \tag{6.6}$$

Proposition 6.4. *Let the conformally parametrized immersion* $X\colon B \to \mathbb{R}^{n+2}$ *be given. Then its Gauss-Osserman map* $\mathfrak{N}$ *is antiholomorphic if and only if* X *is a minimal surface.*

Proof. We compute

$$\overline{\mathfrak{N}}_{\overline{w}} = \mathfrak{N}_w = \frac{1}{2} \triangle X = HW \quad \text{in } B \tag{6.7}$$

with the mean curvature vector H. $\square$

This special analytical behaviour of $\mathfrak{N}$ enabled Osserman in [130], Theorem 12.1 to establish the following characterisitic of the Gauss-Osserman map of minimal surfaces.

Theorem 6.2. *A complete minimal surface is either a plane, or its image under the Gauss-Osserman map* $\mathfrak{N}$ *approaches arbitrarily closely every hyperplane in* $\mathbb{C}P^{n+1}$. *The normals to a complete regular minimal surface in* $\mathbb{R}^{n+2}$ *are everywhere dense unless the surface is a plane.*

6.5 Fubini-Study metric and the total curvature

The so-called Fubini-Study metric is closely related to the Gauss-Osserman map $\mathfrak{N}$. To illustrate this connection we consider a curve $Z = Z(t)$ in $\mathbb{C}P^{n+1}$, and define the *Fubini-Study metric* as

$$\left(\frac{d\hat{s}}{dt}\right)^2 := 2\,\frac{|Z\wedge Z'|^2}{|Z|^4} = 2\,\frac{\sum\limits_{1\le j<k\le n+2} |z_j z_k' - z_k z_j'|^2}{\left(\sum\limits_{j=1}^{n+2} |z_j|^2\right)^2} \tag{6.8}$$

where Z' denotes the derivative w.r.t. the parameter t. This line element $d\hat{s}$ replaces the spherical line element of a surfaces in $\mathbb{R}^3$ as follows (see Osserman [130]).

Proposition 6.5. *Let the conformally parametrized minimal surface $X\colon B \to \mathbb{R}^{n+2}$ be given. Then it holds*

$$\left(\frac{d\hat{s}}{dt}\right)^2 = (-K)W(u'^2+v'^2) = (-K)\left(\frac{ds}{dt}\right)^2 \tag{6.9}$$

for the curve $\mathfrak{N}(t) = X_u(t) + iX_v(t)$.

In other words, the Gauss curvature K of the minimal surface X is the negative of the area magnification under $\mathfrak{N}$, i.e. setting $\widehat{W} = (-K)W$, we obtain

$$K = -\frac{\widehat{W}}{W} = -\frac{d\hat{s}^2}{ds^2} \quad \text{in } B. \tag{6.10}$$

Proof. We must evaluate

$$\begin{aligned}
\mathfrak{N}\wedge\mathfrak{N}' &= (X_u + iX_v)\wedge\left\{X_{uu}u' + X_{uv}v' + i(X_{uv}u' + X_{vv}v')\right\} \\
&= X_u\wedge X_{uu}u' + X_u\wedge X_{uv}v' - X_v\wedge X_{uv}u' - X_v\wedge X_{vv}v' \\
&\quad + i\left\{X_u\wedge X_{uv}u' + X_u\wedge X_{vv}v' + X_v\wedge X_{uu}u' + X_v\wedge X_{uv}v'\right\}.
\end{aligned}$$

The corresponding real part follows from

$$\begin{aligned}
\mathrm{Re}\,(\mathfrak{N}\wedge\mathfrak{N}') &= \left\{\Gamma_{11}^2 u' + \Gamma_{12}^2 v' + \Gamma_{12}^1 u' + \Gamma_{22}^1 v'\right\} X_u\wedge X_v \\
&\quad + \sum_{\sigma=1}^n (L_{\sigma,11}u' + L_{\sigma,12}v')X_u\wedge N_\sigma - \sum_{\sigma=1}^n (L_{\sigma,12}u' + L_{\sigma,22}v')X_v\wedge N_\sigma \\
&= \sum_{\sigma=1}^n (L_{\sigma,11}u' + L_{\sigma,12}v')X_u\wedge N_\sigma - \sum_{\sigma=1}^n (L_{\sigma,12}u' + L_{\sigma,22}v')X_v\wedge N_\sigma
\end{aligned}$$

taking the Gauss equations (2.2) together with (2.6) into account.

Thus, using $L_{\sigma,11} + L_{\sigma,22} = 0$ we have

$$|\mathrm{Re}\,(\mathfrak{N}\wedge\mathfrak{N}')|^2 = W\sum_{\sigma=1}^{n}(L^2_{\sigma,11}u'^2 + L^2_{\sigma,12}u'^2 + L^2_{\sigma,12}v'^2 + L^2_{\sigma,22}v'^2 + \ldots$$

$$\ldots + 2L_{\sigma,11}L_{\sigma,12}u'v' + 2L_{\sigma,12}L_{\sigma,22}u'v')$$

$$= W\sum_{\sigma=1}^{n}(L^2_{\sigma,11}u'^2 + L^2_{\sigma,12}u'^2 + L^2_{\sigma,12}v'^2 + L^2_{\sigma,22}v'^2)\,.$$

Analogously, for the imaginary part we get

$$\mathrm{Im}\,(\mathfrak{N}\wedge\mathfrak{N}') = \sum_{\sigma=1}^{n}(L_{\sigma,12}u' + L_{\sigma,22}v')X_u\wedge N_\sigma + \sum_{\sigma=1}^{n}(L_{\sigma,11}u' + L_{\sigma,22}u')X_v\wedge N_\sigma\,,$$

and therefore it holds

$$|\mathrm{Im}\,(\mathfrak{N}\wedge\mathfrak{N}')|^2 = W\sum_{\sigma=1}^{n}(L^2_{\sigma,12}u'^2 + L^2_{\sigma,22}v'^2 + L^2_{\sigma,11}u'^2 + L^2_{\sigma,12}v'^2).$$

Adding up both terms we arrive at (use $L^2_{\sigma,11} = L^2_{\sigma,22}$)

$$|\mathfrak{N}\wedge\mathfrak{N}'|^2 = W\sum_{\sigma=1}^{n}(L^2_{\sigma,11} + 2L^2_{\sigma,12} + L^2_{\sigma,22})(u'^2 + v'^2)$$

$$= W\sum_{\sigma=1}^{n}\Big\{(L_{\sigma,11} + L_{\sigma,22})^2 - 2(L_{\sigma,11}L_{\sigma,22} - L^2_{\sigma,12})\Big\}(u'^2 + v'^2)$$

$$= 2(-K)W^3(u'^2 + v'^2).$$

Moreover we have

$$|\mathfrak{N}|^4 = |X_u + iX_v|^4 = (2W)^2 = 4W^2$$

which finally implies

$$\left(\frac{d\hat{s}}{dt}\right)^2 = 2\,\frac{2(-K)W^3}{4W^2}\,(u'^2 + v'^2) = (-K)W(u'^2 + v'^2)\,.$$

This proves the statement. $\square$

From $\widehat{W} = (-K)W$ we immediately infer Osserman's result from [130])

Corollary 6.2. *The total curvature of the minimal surface X is the negative of the area $\widehat{\mathscr{A}}$ of the image w.r.t. the metric (6.8), i.e.*

$$\widehat{\mathscr{A}} = \iint_B (-K)W\,dudv. \tag{6.11}$$

This precisely generalizes the situation in $\mathbb{R}^3$ where the curvatura integra equals the image of the spherical mapping if we count multiplicities which eventually arise from spherical branch points.

This leads us to the next remark: The Fubini-Study metric $d\hat{s}^2$ for minimal surfaces is singular at points with $K=0$. But due to Corollary 5.2, in every compact set $\Omega \subset\subset \mathbb{R}$ there are only finitely many singularities unless X is flat.

We want to prove an upper bound for the Gaussian curvature $\widehat{K}$ of the Fubini-Study metric $d\hat{s}^2$ for a minimal surface if its normal bundle is flat. Hoffman and Osserman in [93] have already shown that it holds

$$\widehat{K} \le 2$$

independently of the geometry of the normal bundle of the minimal surface. For a proof we want to refer the reader to this paper.

In case of flat normal bundles we can prover stronger estimates, and that is the purpose of our following considerations.

Lemma 6.2. *Let $X\colon B \to \mathbb{R}^{n+2}$ be a conformally parametrized minimal surface. With a positive number $\kappa_0 > 0$ suppose that*

$$\chi := \kappa_0 - K > 0.$$

Furthermore let $\widehat{K}$ denote the Gaussian curvature of the new metric with coefficients $\widehat{g}_{ij} := \chi g_{ij}$. Then there hold

$$\chi\widehat{K} = K - \frac{1}{2W}\,\Delta\log\chi,$$

and therefore, using Wirtinger symbols,

$$\chi^3\widehat{K} = \chi^2 K + \frac{2}{W}\,(\chi_w\chi_{\overline{w}} - \chi\chi_{w\overline{w}}).$$

Proof. Note that (3.11) together with $\widehat{W} = \chi W$ proves the first identity, namely

$$\begin{aligned}\chi\widehat{K} &= -\frac{1}{W}\,\Delta\log\sqrt{\chi W} = -\frac{1}{W}\,\Delta\log\sqrt{W} - \frac{1}{W}\,\Delta\log\sqrt{\chi}\\ &= K - \frac{1}{2W}\,\Delta\log\chi.\end{aligned}$$

Next we compute

$$\triangle\log\chi = \frac{1}{\chi}\,\triangle\chi - \frac{|\nabla\chi|^2}{\chi^2} = \frac{4}{\chi}\,\chi_{w\overline{w}} - \frac{4}{\chi^2}\,\chi_w\chi_{\overline{w}}.$$

Inserting the right hand side from here into our first identity shows the lemma. □

Lemma 6.3. *It holds*

$$K = \frac{2}{W^2}\left\{\frac{W_w W_{\overline{w}}}{W} - W_{w\overline{w}}\right\}.$$

Proof. We compute

$$K = -\frac{1}{2W}\triangle \log W = -\frac{1}{2W}\left\{\frac{\triangle W}{W} - \frac{|\nabla W|^2}{W^2}\right\} = \frac{2}{W^2}\left\{\frac{W_w W_{\overline{w}}}{W} - W_{w\overline{w}}\right\}$$

proving the statement. □

Now we come to the announced estimate of $\widehat{K}$.

Theorem 6.3. *Let $X\colon B \to \mathbb{R}^{n+2}$ be a conformally parametrized minimal surface with flat normal bundle. Then the Gauss curvature $\widehat{K}$ of the Fubini-Study metric $d\hat{s}^2$ satisfies*

$$\widehat{K} \le 1 \quad \textit{in } B.$$

Proof. Let N be a torsionfree ONF. For the complex-valued Hopf functions $\mathscr{H}_\sigma$ there hold (see also section 5.7)

$$|\mathscr{H}_\sigma|^2 = 4(-K_\sigma)W^2 \quad \text{resp.} \quad \sum_{\sigma=1}^{n}|\mathscr{H}_\sigma|^2 = 4(-K)W^2$$

with the classical Gauss curvature K. Therefore we have

$$\chi = \frac{1}{4W^2}\sum_{\sigma=1}^{n}|\mathscr{H}_\sigma|^2 + \kappa_0$$

for the function $\chi = \kappa_0 - K$. For complex-valued vectors $v = (v_1,\ldots,v_n) \in \mathbb{C}^n$ and $w = (w_1,\ldots,w_n) \in \mathbb{C}^n$ we introduce the following abbreviation

$$w \star v = v \star w := \sum_{\sigma=1}^{n} v_\sigma w_\sigma\,, \quad \|v\|^2 := v \star \overline{v} = \sum_{\sigma=1}^{n} v_\sigma \overline{v}_\sigma = \sum_{\sigma=1}^{n}|v_\sigma|^2\,.$$

Thus, using the Hopf vector $\mathscr{H} = (\mathscr{H}_1,\ldots,\mathscr{H}_n) \in \mathbb{C}^n$ we can rewrite the previous formula to get

$$\chi = \frac{1}{4W^2}\mathscr{H} \star \overline{\mathscr{H}} + \kappa_0 = \frac{1}{4W^2}\|\mathscr{H}\|^2 + \kappa_0\,.$$

Together with $\mathscr{H}_{\overline{w}} = 0$ from (5.15) we calculate

$$\chi_w = \frac{1}{4W^2}\overline{\mathscr{H}} \star \mathscr{H}_w - \frac{W_w}{2W^3}\|\mathscr{H}\|^2 = \frac{1}{4W^2}\overline{\mathscr{H}} \star \mathscr{H}_w - \frac{2W_w}{W}(\chi - \kappa_0),$$

$$\chi_{\overline{w}} = \frac{1}{4W^2}\mathscr{H} \star \overline{\mathscr{H}}_{\overline{w}} - \frac{W_{\overline{w}}}{2W^3}\|\mathscr{H}\|^2 = \frac{1}{4W^2}\mathscr{H} \star \overline{\mathscr{H}}_{\overline{w}} - \frac{2W_{\overline{w}}}{W}(\chi - \kappa_0)$$

for the first complex derivatives.

From here we further induce

$$\chi_{w\overline{w}} = -\frac{1}{2W^3}\left\{W_{\overline{w}}\overline{\mathscr{H}}\star\mathscr{H}_w + W_w\,\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}\right\} + \frac{1}{4W^2}\,\|\mathscr{H}_w\|^2$$
$$-\frac{W_{w\overline{w}}}{2W^3}\,\|\mathscr{H}\|^2 + \frac{3W_wW_{\overline{w}}}{2W^4}\,\|\mathscr{H}\|^2.$$

It follows

$$\begin{aligned}
&\chi_w\chi_{\overline{w}} - \chi\chi_{w\overline{w}}\\
&= \frac{1}{16W^4}\,(\overline{\mathscr{H}}\star\mathscr{H}_w)(\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}})\\
&\quad -\frac{1}{8W^5}\left\{W_{\overline{w}}\overline{\mathscr{H}}\star\mathscr{H}_w + W_w\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}\right\}\|\mathscr{H}\|^2\\
&\quad +\frac{W_wW_{\overline{w}}}{4W^6}\,\|\mathscr{H}\|^4 - \frac{\|\mathscr{H}\|^2\|\mathscr{H}_w\|^2}{16W^4} - \frac{\kappa_0\|\mathscr{H}_w\|^2}{4W^2}\\
&\quad +\left\{W_{\overline{w}}\overline{\mathscr{H}}\star\mathscr{H}_w + W_w\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}\right\}\left(\frac{\|\mathscr{H}\|^2}{8W^5} + \frac{\kappa_0}{2W^3}\right)\\
&\quad -\left(\frac{\|\mathscr{H}\|^2}{4W^2} + \kappa_0\right)\left(\frac{3W_wW_{\overline{w}}}{2W^4} - \frac{W_{w\overline{w}}}{2W^3}\right)\|\mathscr{H}\|^2\\
&= \frac{1}{16W^4}\left\{(\overline{\mathscr{H}}\star\mathscr{H}_w)(\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}) - \|\mathscr{H}\|^2\|\mathscr{H}_w\|^2\right\}\\
&\quad -\frac{W_wW_{\overline{w}}}{8W^6}\,\|\mathscr{H}\|^4 + \frac{W_{w\overline{w}}}{8W^5}\,\|\mathscr{H}\|^4 - \frac{W_wW_{\overline{w}}\kappa_0\|\mathscr{H}\|^2}{2W^4} + \frac{W_{w\overline{w}}\kappa_0\|\mathscr{H}\|^2}{2W^3}\\
&\quad -\frac{\kappa_0}{4W^2}\left\{\|\mathscr{H}_w\|^2 - \frac{2}{W}\,W_{\overline{w}}\overline{\mathscr{H}}\star\mathscr{H}_w - \frac{2}{W}\,W_w\,\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}} + 4\,\frac{W_wW_{\overline{w}}}{W^2}\,\|\mathscr{H}\|^2\right\}\\
&= \frac{1}{16W^4}\left\{(\overline{\mathscr{H}}\star\mathscr{H}_w)(\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}) - \|\mathscr{H}\|^2\|\mathscr{H}_w\|^2\right\}\\
&\quad -\frac{\|\mathscr{H}\|^2}{2W^3}\left\{\frac{W_wW_{\overline{w}}}{W} - W_{w\overline{w}}\right\}\left\{\kappa_0 + \frac{\|\mathscr{H}\|^2}{4W^2}\right\}\\
&\quad -\frac{\kappa_0}{4W^2}\left\|\mathscr{H}_w - \frac{2}{W}\,W_w\mathscr{H}\right\|^2.
\end{aligned}$$

Thus we arrive at the identity

$$\chi_w\chi_{\overline{w}} - \chi\chi_{w\overline{w}} = \frac{1}{16W^4}\left\{(\overline{\mathscr{H}}\star\mathscr{H}_w)(\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}) - \|\mathscr{H}\|^2\|\mathscr{H}_w\|^2\right\}$$
$$-\frac{\|\mathscr{H}\|^2K}{4W}\chi - \frac{\kappa_0}{4W^2}\left\|\mathscr{H}_w - \frac{2}{W}W_w\mathscr{H}\right\|^2.$$

Notice that

$$-\frac{\|\mathscr{H}\|^2K}{4W} = \frac{\chi^3}{2}W - \frac{\chi^2}{2}KW + \chi\left(\kappa_0 - \frac{3}{2}\chi\right)\kappa_0 W$$
$$\leq \frac{\chi^3}{2}W - \frac{\chi^2}{2}KW.$$

On the other hand, we have

$$(\overline{\mathscr{H}}\star\mathscr{H}_w)(\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}) - \|\mathscr{H}\|^2\|\mathscr{H}_w\|^2$$
$$= \left(\sum_{\sigma=1}^n \overline{\mathscr{H}}_\sigma\mathscr{H}_{\sigma,w}\right)\left(\sum_{\sigma=1}^n \mathscr{H}_\sigma\overline{\mathscr{H}}_{\sigma,\overline{w}}\right) - \left(\sum_{\sigma=1}^n |\mathscr{H}_\sigma|^2\right)\left(\sum_{\sigma=1}^n |\mathscr{H}_{\sigma,w}|^2\right)$$
$$= \left|\sum_{\sigma=1}^n \overline{\mathscr{H}}_\sigma\mathscr{H}_{\sigma,w}\right|^2 - \left(\sum_{\sigma=1}^n |\mathscr{H}_\sigma|^2\right)\left(\sum_{\sigma=1}^n |\mathscr{H}_{\sigma,w}|^2\right) \leq 0.$$

Therefore we conclude

$$\chi^3\widehat{K} = \chi^2K + \chi^3 - \chi^2K - \frac{\kappa_0}{2W^3}\left\|\mathscr{H}_w - \frac{2}{W}W_w\mathscr{H}\right\|^2$$
$$+\frac{1}{8W^5}\left\{(\overline{\mathscr{H}}\star\mathscr{H}_w)(\mathscr{H}\star\overline{\mathscr{H}}_{\overline{w}}) - \|\mathscr{H}\|^2\|\mathscr{H}_w\|^2\right\}$$
$$\leq \chi^3.$$

The statement follows. □

The proof presented here is motivated by Ruchert's considerations from [137]. We will apply Ruchert's ingenious methods again in section 13.6.

6.6 Minimal surfaces with constant curvature $\widehat{K}$

We want to conclude this chapter with some interesting characterizations of the shape of minimal surfaces in $\mathbb{R}^4$ with constant curvature $\widehat{K}$ which we take again from Hoffman and Osserman [93].

Theorem 6.4. *There hold the following alternatives.*

1. *The generalized Gauss image of the minimal immersion* $X: B \to \mathbb{R}^4$ *has constant Gauss curvature* $\widehat{K} \equiv c$. *Then,* $c = 1$ *or* $c = 2$, *and*
   - $\widehat{K} \equiv 1$ *if and only if* $X(B)$ *lies fully in some* $\mathbb{R}^3$;
   - $\widehat{K} \equiv 2$ *if and only if* $X(B)$ *is a complex curve lying fully in* $\mathbb{C}^2$, *represented by* $\mathbb{R}^4$ *endowed with some orthogonal complex structure.*
2. *If the generalized Gauss image of the minimal immersion* $X: B \to \mathbb{R}^{n+2}$ *has constant Gauss curvature* $\widehat{K} \equiv 2$, *then* $X(B)$ *lies in an affine subspace* $\mathbb{R}^4 \subset \mathbb{R}^{n+2}$.

Part II
Variational Problems

Chapter 7
Normal Coulomb frames in $\mathbb{R}^4$

In this chapter we focus on orthonormal frames in the normal space of immersions in $\mathbb{R}^4$ *which are critical for a functional of total torsion. This generalizes the concept of parallel normal frames for immersions with non-flat normal bundles.*

7.1 The total torsion

In sections 5.3 and 5.4 we have introduced the *torsion T of an ONF N* in the form

$$T = \frac{1}{2}\sum_{i,j=1}^{2}\sum_{\sigma,\vartheta=1}^{n} g^{ij}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}$$

as well as the following *functional of total torsion*

$$\mathscr{T}_X[N] = \iint_B TW\,dudv = \frac{1}{2}\sum_{i,j=1}^{2}\sum_{\sigma,\vartheta=1}^{n}\iint_B g^{ij}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}W\,dudv.$$

This functional does not depend on the choice of the parametrization. *But it depends on the choice of the ONF N*. In particular, if there exist a torsionfree ONF N then obviously $\mathscr{T}_X[N] = 0$. Otherwise it is always greater then zero.

We want to construct orthonormal normal frames N which are minimal for $\mathscr{T}[N]$.

7.2 Curves in $\mathbb{R}^3$

To illustrate the underlying idea we want to start with constructing *torsionfree normal frames for one-dimensional curves in* $\mathbb{R}^3$.

Let the regular and arc-length parametrized curve $c = c(s)$ be given together with the orthogonal moving 3-frame

$$\{t(s), n(s), b(s)\}, \quad s \in I \subset \mathbb{R},$$

consisting of the unit tangential vector $t(s)$, the unit normal vector $n(s)$, and the unit binormal vector $b(s)$.

These vectors are given as follows

$$t(s) = c'(s), \quad n(s) = \frac{c''(s)}{|c''(s)|}, \quad b(s) = t(s) \times n(s)$$

under the assumptions $c'(s) \neq 0$ and $c''(s) \neq 0$ for the first and second derivatives of the mapping $c(s)$. The curve's curvature $\kappa(s)$ and torsion $\tau(s)$ are

$$\kappa(s) = |t'(s)|^2, \quad \tau(s) = n'(s) \cdot b(s) = -b'(s) \cdot n(s).$$

Consider now a new ONF $(\widetilde{n}, \widetilde{b})$ resulting from

$$\widetilde{n}(s) := \cos\varphi\, n(s) + \sin\varphi\, b(s), \quad \widetilde{b}(s) := -\sin\varphi\, n(s) + \cos\varphi\, b(s)$$

with a rotation angle φ. Again there hold

$$|\widetilde{n}(s)| = 1, \quad |\widetilde{b}(s)| = 1, \quad \widetilde{n}(s) \cdot \widetilde{b}(s) = 0, \quad \widetilde{n}(s) \cdot t(s) = 0, \quad \widetilde{b}(s) \cdot t(s) = 0$$

for all $s \in I$. The torsion w.r.t. this new frame, i.e.

$$\widetilde{\tau}(s) := \widetilde{n}'(s) \cdot \widetilde{b}(s),$$

can be computed explicitely as follows

$$\begin{aligned} \widetilde{\tau} &= (-\varphi' \sin\varphi\, n + \cos\varphi\, n' + \varphi' \cos\varphi\, b + \sin\varphi\, b') \cdot (-\sin\varphi\, n + \cos\varphi\, b) \\ &= \varphi' \sin^2\varphi + \cos^2\varphi\, n' \cdot b + \varphi' \cos^2\varphi - \sin^2\varphi\, b' \cdot n \\ &= \varphi' + n' \cdot b. \end{aligned}$$

Thus we arrive at the *transformation formula for the two torsions*

$$\varphi'(s) = \widetilde{\tau}(s) - \tau(s).$$

In particular, solving the ordinary differential equation $\varphi'(s) = -\tau(s)$, *expressing the property* $\widetilde{\tau}(s) \equiv 0$, *yields a rotation angle function* $\varphi = \varphi(s)$ *which transforms the given ONF* (n, b) *into a new torsionfree ONF* $(\widetilde{n}, \widetilde{b})$.

Such frames are also called *parallel* for their *Frenét equations do not contain normal components.*

The following considerations are devoted to the construction of torsionfree orthonormal normal frames for surfaces in space $\mathbb{R}^4$. This problem becomes more intricate because it will turn out that such a construction reduces to solving a partial differential equation instead of an ordinary differenial equation.

7.3 Coulomb ONF

Let the immersion $X\colon B\to\mathbb{R}^4$ together with an ONF $N=(N_1,N_2)$ be given. We set

$$\widetilde{N}_1=\cos\varphi\,N_1+\sin\varphi\,N_2\,,\quad \widetilde{N}_2=-\sin\varphi\,N_1+\cos\varphi N_2 \tag{7.1}$$

for a new ONF $(\widetilde{N}_1,\widetilde{N}_2)$ resulting from the action of the $SO(2)$-group realized by a rotation angle $\varphi\in[0,2\pi]$. Then the situation now is comparable to the situation of onedimensional curves in $\mathbb{R}^3$.

Lemma 7.1. *The torsion coefficients of the ONF $\{\widetilde{N}_1,\widetilde{N}_2\}$ are given by*

$$\widetilde{T}^2_{1,1}=T^2_{1,1}+\varphi_u\,,\quad \widetilde{T}^2_{1,2}=T^2_{1,2}+\varphi_v\,. \tag{7.2}$$

Proof. Taking account of $N_\sigma\cdot N_{\sigma,u^i}=0$ we compute

$$\begin{aligned}\widetilde{T}^2_{1,1}&=(-\varphi_u\sin\varphi\,N_1+\cos\varphi\,N_{1,u}+\varphi_u\cos\varphi\,N_2+\sin\varphi\,N_{2,u})\cdot\ldots\\&\qquad\ldots\cdot(-\sin\varphi\,N_1+\cos\varphi\,N_2)\\&=\varphi_u\sin^2\varphi-\sin^2\varphi\,T^1_{2,1}+\cos^2\varphi\,T^2_{1,1}+\varphi_u\cos^2\varphi\\&=T^2_{1,1}+\varphi_u\,,\end{aligned}$$

and analoguously it follows $\widetilde{T}^2_{1,2}=T^2_{1,2}+\varphi_v$. □

The remaining components $T^\sigma_{\sigma,i}$ vanish identically.

We are particularly interested in the difference

$$\triangle\mathscr{T}_X:=\mathscr{T}_X[\widetilde{N}]-\mathscr{T}_X[N],$$

i.e. in critical points of the functional of total torsion. For this purpose we introduce conformal parameters $(u,v)\in B$ and calculate

$$\begin{aligned}\triangle\mathscr{T}_X&=\iint\limits_B|\nabla\varphi|^2\,dudv+2\iint\limits_B(T^2_{1,1}\varphi_u+T^2_{1,2}\varphi_v)\,dudv\\&=\iint\limits_B|\nabla\varphi|^2\,dudv+2\int\limits_{\partial B}(T^2_{1,1},T^2_{1,2})\cdot\nu\,\varphi\,ds-2\iint\limits_B\operatorname{div}(T^2_{1,1},T^2_{1,2})\varphi\,dudv\end{aligned} \tag{7.3}$$

by partial integration, ν being the outer unit normal vector ν at the boundary ∂B.

In general, the right hand side of this identity does not vanishes. In this way we immediately obtain the Euler-Lagrange equation for a $\mathscr{T}_X$-critical ONF.

Proposition 7.1. *Let the ONF (N_1,N_2) of the conformally parametrized immersion $X\colon B\to\mathbb{R}^4$ be critical for the functional of total torsion $\mathscr{T}_X[N]$. Then there hold*

$$\operatorname{div}(T^2_{1,1},T^2_{1,2})=0\quad\text{in }\mathring{B},\quad (T^2_{1,1},T^2_{1,2})\cdot\nu=0\quad\text{on }\partial B. \tag{7.4}$$

This leads us directly to the following

Definition 7.1. An ONF N satisfying (7.4) is called a *normal Coulomb frame.*

The terminology *Coulomb frame* was suggested by F. Helein. It is motivated by the similarity of (7.4) to differential equations from electromagnetism where a vector field $A \in \mathbb{R}^3$ is called *Coulomb gauged* if it satisfies $\operatorname{div} A = 0$ with the spatial divergence operator.

7.4 Construction of normal Coulomb frames

We want to follow up on the question of *constructing a $\mathcal{T}_X[N]$-critical frame (N_1, N_2) from a given ONF $(\widetilde{N}_1, \widetilde{N}_2)$.*

If (N_1, N_2) is actually critical for $\mathcal{T}_X[N]$ then we know

$$
\begin{aligned}
0 &= \operatorname{div}(T_{1,1}^2, T_{1,2}^2) = \operatorname{div}(\widetilde{T}_{1,1}^2 - \varphi_u, \widetilde{T}_{1,2}^2 - \varphi_v) \quad \text{in } \mathring{B}, \\
0 &= (T_{1,1}^2, T_{1,2}^2) \cdot \nu = (\widetilde{T}_{1,1}^2 - \varphi_u, \widetilde{T}_{1,2}^2 - \varphi_v) \cdot \nu \quad \text{on } \partial B,
\end{aligned}
$$

by virtue of (7.2) and (7.4). This implies our next result.

Proposition 7.2. *Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^4$ be given. Its ONF $(\widetilde{N}_1, \widetilde{N}_2)$ transforms into a $\mathcal{T}_X[N]$-critical ONF (N_1, N_2) by means of (7.2) if and only if the new torsion coefficients satisfy the following Neumann boundary value problem*

$$
\begin{aligned}
\Delta\varphi &= \operatorname{div}(\widetilde{T}_{1,1}^2, \widetilde{T}_{1,2}^2) \quad \textit{in } \mathring{B}, \\
\frac{\partial\varphi}{\partial\nu} &= (\widetilde{T}_{1,1}^2, \widetilde{T}_{1,2}^2) \cdot \nu \quad \textit{on } \partial B
\end{aligned}
\tag{7.5}
$$

with a critical rotation angle $\varphi = \varphi(u,v)$.

It is well known that the Neumann problem

$$
\Delta\varphi = f \quad \text{in } \mathring{B}, \qquad \frac{\partial\varphi}{\partial\nu} = g \quad \text{on } \partial B
$$

has a solution if and only if the integrability condition

$$
\iint\limits_B f\,du dv = \int\limits_{\partial B} g\,ds
$$

holds true. Obviously this condition is fulfilled in our situation, and thus there actually exist a normal Coulomb frame N. Note that such a frame *is not uniquely determined* because *a rotation of the whole normal Coulomb frame N about an arbitrary global constant angle φ_0 does not affect the Euler-Lagrange equation.*

7.5 Minimality of normal Coulomb frames

Although normal Coulomb frames are not uniquely determined we can prove that *they actually minimize the functional of total torsion.*

For this purpose let $N = (N_1, N_2)$ be a $\mathscr{T}_X[N]$-critical ONF. Applying the above $SO(2)$-action with a rotation angle φ we infer

$$\mathscr{T}[\widetilde{N}] = \mathscr{T}[N] + 2\iint\limits_B |\nabla\varphi|^2\,dudv$$

taking (7.3) together with the Euler-Lagrange equation (7.4) into account.

Proposition 7.3. *A normal Coulomb frame N of the immersion $X\colon B \to \mathbb{R}^4$ minimizes the functional $\mathscr{T}_X[N]$ of total torsion, i.e. it holds*

$$\mathscr{T}[N] \le \mathscr{T}[\widetilde{N}]$$

for all normal frames $\widetilde{N}$ resulting from (7.1).

7.6 A torsion estimates via the maximum principle

Let $(u,v) \in B$ be conformal parameters. We want to establish upper bounds for the torsion coefficients of normal Coulomb frames of $X\colon B \to \mathbb{R}^4$.

Assume first that the surface X has flat normal bundle. Then our considerations from section 3.7 yield

$$S_{1,12}^2 = \partial_v T_{1,1}^2 - \partial_u T_{1,2}^2 = \operatorname{div}(-T_{1,2}^2, T_{1,1}^2) \equiv 0 \quad \text{in } B \tag{7.6}$$

for the only non-trivial component of the curvature tensor of the normal bundle. Furthermore, from the Euler-Lagrange equation (7.4) we infer that for a $\mathscr{T}_X[N]$-critical ONF N, *the vector field $(-T_{1,2}^2, T_{1,1}^2)$ is parallel to the outer unit normal vector ν along ∂B.* Thus partial integration gives us

$$\iint\limits_B S_{1,12}^2\,dudv = \int\limits_{\partial B} (-T_{1,2}^2, T_{1,1}^2)\cdot\nu\,ds = \pm\int\limits_{\partial B}\sqrt{(T_{1,1}^2)^2 + (T_{1,2}^2)^2}\,ds.$$

In particular, in case of flat normal bundles with $S_{1,12}^2 \equiv 0$ we find

$$T_{\sigma,i}^{\vartheta} \equiv 0 \quad \text{on } \partial B$$

for all $i = 1,2$ and $\sigma,\vartheta = 1,2$.

On the other hand, differentiating (7.6) and taking the Euler-Lagrange equation (7.4) into account, we arrive at

$$\Delta T_{1,1}^2 = \frac{\partial}{\partial v} S_{1,12}^2 = 0\,, \quad \Delta T_{1,2}^2 = -\frac{\partial}{\partial u} S_{1,12}^2 = 0$$

for flat normal bundles. *Therefore $T_{1,1}^2$ and $T_{1,2}^2$ are harmonic functions, and the maximum principle implies*

$$T_{\sigma,i}^{\vartheta} \equiv 0 \quad \text{in } B \quad \text{for all } i = 1,2 \text{ and } \sigma, \vartheta = 1,2. \tag{7.7}$$

This now holds true for all parametrizations of class $\mathfrak{P}$.

Theorem 7.1. *A normal Coulomb frame N of an immersion $X\colon B \to \mathbb{R}^4$ with flat normal bundle is free of torsion.*

In summary we have proved existence of regular normal Coulomb frames, and if additionally the normal bundle of the surface is flat then those frames are free of torsion. Existence results concerning such situations in case of manifolds immersed in Euclidean spaces of higher dimension can be found e.g. in Chen [27].

Next we want to consider the case of *non-flat normal bundles*. Due to the Euler-Lagrange equation (7.4), the torsion vector $(T_{1,1}^2, T_{1,2}^2)$ of a normal Coulomb frame is divergence-free. Thus the differential 1-form

$$\omega := -T_{1,2}^2\,du + T_{1,1}^2\,dv$$

is closed, i.e.

$$d\omega = \partial_u T_{1,1}^2\,du \wedge dv - \partial_v T_{1,2}^2\,dv \wedge du = \operatorname{div}(T_{1,1}^2, T_{1,2}^2)\,du \wedge dv = 0.$$

Now from Poincare's lemma we infer the existence of a C^3-regular function τ with the property

$$d\tau = \tau_u\,du + \tau_v\,du = \omega, \quad \text{i.e. } \nabla\tau = (-T_{1,2}^2, T_{1,1}^2).$$

Differentiating, taking again $S_{1,12}^2 = \partial_v T_{1,1}^2 - \partial_u T_{1,2}^2$ into account, leads to the homogeneous boundary value problem

$$\triangle\tau = S_{1,12}^2 \quad \text{in } \mathring{B}, \quad \tau = 0 \quad \text{on } \partial B. \tag{7.8}$$

To justify the homogeneous boundary values note that $\nabla\tau \cdot (-v,u) = 0$ on ∂B since $(-v,u) \in \partial B$ represents a tangential vector. We conclude $\tau = \text{const}$ on ∂B. But τ is only defined up to a constant which can be chosen such that (7.8) is true.

Next we want to establish bounds for τ and its gradient.

First use Poisson's representation formula for the solution τ of the foregoing Dirichlet boundary value problem, i.e.

$$\tau(w) = \iint\limits_B \Phi(\zeta;w) S_{1,12}^2(\zeta)\,d\xi\,d\eta, \quad \zeta = (\xi,\eta), \tag{7.9}$$

with the non-positive Green's function

$$\Phi(\zeta;w) := \frac{1}{2\pi}\log\left|\frac{\zeta - w}{1-\overline{w}\zeta}\right|, \quad \zeta \neq w, \tag{7.10}$$

for the Laplace operator $\triangle$ (see e.g. Sauvigny [143], chapter VIII, §1). Note that

$$\psi(w) = \frac{|w|^2-1}{4} \quad \text{solves} \quad \triangle\psi = 1 \text{ in } \mathring{B} \text{ and } \psi = 0 \text{ on } \partial B.$$

Therefore Poisson's representation formula yields

$$\iint\limits_B |\Phi(\zeta;w)|\,d\xi\,d\eta = \frac{1-|w|^2}{4} \le \frac{1}{4}\,.$$

Lemma 7.2. *The solution τ from (7.8), (7.9) satisfies*

$$|\tau(w)| \le \frac{1}{4}\,\|S_{1,12}^2\|_{C^0(B)}\,. \tag{7.11}$$

This is nothing else but a maximum principle for the above homogeneous boundary value problem for the integral function τ.

Potential theoretic estimates for the Laplacian (see e.g. Sauvigny [143], chapter IX, §4, Satz 1) ensure finally the existence of a real constant $C = C(\alpha)$ such that

$$\|\tau\|_{C^{2+\alpha}(B)} \le C(\alpha)\|S_{1,12}^2\|_{C^\alpha(B)} \tag{7.12}$$

for all $\alpha \in (0,1)$. But the gradient of τ codifies the torsion coefficients. Thus we have proved the main result of this section.

Theorem 7.2. *Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^4$ with normal bundle of curvature $S_{1,12}^2$ be given. Then the torsion coefficients of a normal Coulomb frame $N = (N_1, N_2)$ satisfy*

$$\|T_{\sigma,i}^{\vartheta}\|_{C^{1+\alpha}(B)} \le C(\alpha)\|S_{1,12}^2\|_{C^\alpha(B)} \tag{7.13}$$

for all $\alpha \in (0,1)$ with the constant $C(\alpha)$ from (7.12).

We particularly recover (7.7) for flat normal bundles with $\|S_{1,12}^2\|_{C^\alpha(B)} = 0$.

7.7 A torsion estimate via a Riemann-Hilbert problem

Once again, let us consider (7.4) and (7.6) for normal Coulomb frames

$$\frac{\partial}{\partial u}T_{1,1}^2+\frac{\partial}{\partial v}T_{1,2}^2=0, \quad \frac{\partial}{\partial v}T_{1,1}^2-\frac{\partial}{\partial u}T_{1,2}^2=S_{1,12}^2\,.$$

We immediately obtain

Lemma 7.3. *The complex-valued torsion*

$$\Psi:=T_{1,1}^2-iT_{1,2}^2\in\mathbb{C}$$

solves the non-homogeneous Cauchy-Riemann equation

$$\Psi_{\overline{w}}=\frac{1}{2}\,(\partial_u\Psi+i\partial_v\Psi)=\frac{1}{2}\,(\partial_uT_{1,1}^2+\partial_vT_{1,2}^2)+\frac{i}{2}\,(\partial_vT_{1,1}^2-\partial_uT_{1,2}^2)=\frac{i}{2}\,S_{1,12}^2 \tag{7.14}$$

in the open disc $\mathring{B}$.

Note that *the right hand side here is purely imaginary.* In addition we write the boundary condition from (7.4) as

$$\operatorname{Re}\big[w\Psi(w)\big]=\operatorname{Re}\big[(T_{1,1}^2-iT_{1,2}^2)(u+iv)\big]=T_{1,1}^2u+T_{1,2}^2v=(T_{1,1}^2,T_{1,2}^2)\cdot\nu \tag{7.15}$$

on the boundary curve ∂B with its outer unit normal vector $\nu=(u,v)$.

The relations (7.14) and (7.15) form what is called a *linear Riemann-Hilbert problem* for Ψ. There is a big machinery to treat such mathematical problem using tools from complex analysis. We want to refer the reader e.g. to Begehr and Wen [10], Courant and Hilbert [40], Sauvigny [143], Vekua [161], or Wendland [164] for a first orientation as well as various detailed studies.

Below we construct an explicit solution of our Riemann-Hilbert problem. As it already turns out now this solution is the only possible.

Lemma 7.4. *The Riemann-Hilbert problem (7.14), (7.15) possesses at most one solution $\Psi\in C^1(\mathring{B},\mathbb{C})\cap C^0(B,\mathbb{C})$.*

Proof. Assume there are two such solutions Ψ_1 and Ψ_2. Then we set

$$\Phi(w):=w[\Psi_1(w)-\Psi_2(w)]$$

and compute

$$\Phi_{\overline{w}}=0 \quad \text{in } \mathring{B}, \qquad \operatorname{Re}\Phi=0 \quad \text{on } \partial B.$$

Consequently, it holds $\Phi\equiv ic$ in B with some real constant $c\in\mathbb{R}$. But $\Phi(w)$ is a continuous function, and $\Phi(0)=0$ implies $c=0$. Thus $\Psi_1\equiv\Psi_2$. $\square$

Following Vekua [161] (see also Sauvigny [143]) we will solve our Riemann-Hilbert problem in terms of so-called *generalized analytic functions.* Here come some facts about this important class of complex-valued functions.

For arbitrary $f \in C^1(B,\mathbb{C})$ we define *Cauchy's integral operator* by

$$T_B[f](w) := -\frac{1}{\pi}\iint\limits_B \frac{f(\zeta)}{\zeta - w}\,d\xi\,d\eta, \quad w \in \mathbb{C}. \tag{7.16}$$

Lemma 7.5. *There hold $T_B[f] \in C^1(\mathbb{C}\setminus\partial B)\cap C^0(\mathbb{C})$ as well as*

$$\frac{\partial}{\partial \overline{w}}\,T_B[f](w) = \begin{cases} f(w), & w \in \mathring{B} \\ 0, & w \in \mathbb{C}\setminus B \end{cases}. \tag{7.17}$$

Proof. We want to verify briefly the validity of the stated complex-valued differential equation, see Sauvigny [143], chapter IV, §5. Let $\{G_k\}_{k\in\mathbb{N}}$ be a sequence of open, simply connected and smoothly bounded domains contracting to some point $z_0 \in B$ for $k \to \infty$. Let $|G_k|$ denote the respective areas. Making use of *Cauchy's integral formula*

$$\int\limits_{G_k} \frac{g(w)}{w-\zeta}\,dw = 2\pi i g(\zeta)$$

we compute with the characteristic functions χ_k

$$\begin{aligned}
\frac{1}{2i|G_k|}\int\limits_{\partial G_k} T_B[f](w)\,dw &= \frac{1}{2i|G_k|}\int\limits_{\partial G_k}\left(-\frac{1}{\pi}\iint\limits_B \frac{f(\zeta)}{\zeta-w}\,d\xi\,d\eta\right)dw \\
&= \frac{1}{2\pi i|G_k|}\iint\limits_B \left(f(\zeta)\int\limits_{\partial G_k}\frac{1}{w-\zeta}\,dw\right)d\xi\,d\eta \\
&= \frac{1}{2\pi i|G_k|}\iint\limits_B f(\zeta)\cdot 2\pi i\chi_{G_k}(\zeta)\,d\xi\,d\eta \\
&= \frac{1}{|G_k|}\iint\limits_{G_k} f(\zeta)\,d\xi\,d\eta.
\end{aligned}$$

Now recalling *the integration by parts rule in complex form*

$$\iint\limits_{G_k}\frac{d}{d\overline{w}}\,f(w)\,d\xi\,d\eta = \frac{1}{2i}\int\limits_{\partial G_k} f(z)\,dz$$

we get in the limit

$$\frac{d}{d\overline{w}}\,T_B[f](w) = \lim_{k\to\infty}\frac{1}{2i|G_k|}\int\limits_{\partial G_k} T_B[f](w)\,dw = f(z_0)$$

for all $z_0 \in B$. The statement follows. □

Next we set

$$P_B[f](w) := -\frac{1}{\pi}\iint\limits_B \left\{\frac{f(\zeta)}{\zeta - w} + \frac{\overline{\zeta}\,\overline{f(\zeta)}}{1 - w\overline{\zeta}}\right\} d\xi\, d\eta$$
$$= T_B[f](w) + \frac{1}{w}\,\overline{T_B[wf]\Big(\frac{1}{\overline{w}}\Big)}.$$

Now Satz 1.24 from Vekua [161] states the following

Lemma 7.6. *With the definitions above we have the uniform estimate*

$$\big|P_B[f](w)\big| \le C(p)\|f\|_{L^p(B)}, \quad w \in B,$$

where $p \in (2, +\infty]$, *and* $C(p)$ *is a positive constant dependent only on* p.

Using this result (which remains unproved here) we obtain the following main result of this section.

Theorem 7.3. *Let the conformally parametrized immersion* $X\colon B \to \mathbb{R}^4$ *be given. Then the complex-valued torsion* Ψ *of a normal Coulomb frame* N *satisfies*

$$|\Psi(w)| \le c(p)\|S^2_{1,12}\|_{L^p(B)} \quad \text{for all } w \in B$$

with some positive constant $c(p)$ *and* $p \in (2, +\infty]$.

For flat normal bundles we again verify that a normal Coulomb frame is free of torsion: $T^{\vartheta}_{\sigma,i} \equiv 0$.

Proof. Let us write

$$f := \frac{i}{2}\,S^2_{1,12} \in C^1(B, \mathbb{C})$$

to apply the foregoing results. We claim that the complex-valued torsion Ψ possesses the integral representation

$$\Psi(w) = P_B[f](w) = -\frac{1}{\pi}\iint\limits_B \left\{\frac{f(\zeta)}{\zeta - w} + \frac{\overline{\zeta}\,\overline{f(\zeta)}}{1 - w\overline{\zeta}}\right\} d\xi\, d\eta, \quad w \in \mathring{B}.$$

Then the stated estimate follows at once from the above lemma. First we claim

$$wP_B[f](w) = \frac{1}{\pi}\iint\limits_B f(\zeta)\, d\xi\, d\eta + T_B[wf](w) - \overline{T_B[wf]\Big(\frac{1}{\overline{w}}\Big)}.$$

Let us check this identity:

$$\begin{aligned}
&\frac{1}{\pi}\iint\limits_B f(\zeta)\,d\xi d\eta + T_B[wf](w) - \overline{T_B[wf](\overline{w}^{-1})} \\
&\quad = \frac{1}{\pi}\iint\limits_B f(\zeta)\,d\xi d\eta - \frac{1}{\pi}\iint\limits_B \frac{\zeta f(\zeta)}{\zeta - w}\,d\xi d\eta - \overline{T_B[wf](\overline{w}^{-1})} \\
&\quad = -\frac{w}{\pi}\iint\limits_B \frac{f(\zeta)}{\zeta - w}\,d\xi d\eta + \frac{1}{\pi}\iint\limits_B \frac{\overline{\zeta}\,\overline{f(\zeta)}}{\overline{\zeta} - \frac{1}{w}}\,d\xi d\eta \\
&\quad = -\frac{w}{\pi}\iint\limits_B \frac{f(\zeta)}{\zeta - w}\,d\xi d\eta + \frac{w}{\pi}\iint\limits_B \frac{\overline{\zeta}\,\overline{f(\zeta)}}{\overline{\zeta}\,w - 1}\,d\xi d\eta \\
&\quad = -\frac{w}{\pi}\iint\limits_B \left(\frac{f(\zeta)}{\zeta - w} + \frac{\overline{\zeta}\,\overline{f(\zeta)}}{1 - \overline{\zeta}\,w}\right) d\xi d\eta
\end{aligned}$$

which already shows the stated identity.

Next, taking $f = \frac{i}{2}S_{1,12}^2$ into account, we infer

$$\operatorname{Re}\left\{wP_B[f](w)\right\} = 0, \quad w \in \partial B,$$

what follows from

$$\begin{aligned}
&T_B[\tfrac{1}{2}iwS_{1,12}^2](w) - \overline{T_B[\tfrac{1}{2}iwS_{1,12}^2](\overline{w}^{-1})} \\
&\quad = -\frac{1}{\pi}\iint\limits_B \frac{i}{2}\,\frac{\zeta S_{1,12}^2}{\zeta - w}\,d\xi d\eta + \overline{\frac{1}{\pi}\iint\limits_B \frac{i}{2}\,\frac{\zeta S_{1,12}^2}{\zeta - \overline{w}^{-1}}} \\
&\quad = -\frac{i}{2\pi}\iint\limits_B \left(\frac{\zeta}{\zeta - w} + \frac{\overline{\zeta}}{\overline{\zeta} - \frac{1}{w}}\right) S_{1,12}^2\,d\xi d\eta.
\end{aligned}$$

The entry in the brackets is a real number because $\frac{1}{w} = \frac{\overline{w}}{|w|^2} = \overline{w}$ holds true on the boundary ∂B. Investing additionally

$$\frac{\partial}{\partial \overline{w}} P_B[f](w) = f(w)$$

due to (7.17) and our representation of $P_B[f](w)$ we conclude that $P_B[f](w)$ solves the Riemann-Hilbert problem for Ψ. The previous uniqueness result proves the stated representation. □

Note that our proof relies crucially on the fact *that $f = \frac{i}{2}S_{1,12}^2$ is purely imaginary!*

7.8 Estimates for the total torsion

The previous results allow us immediately to establish lower and upper bounds for the total torsion $\mathscr{T}_X[N]$ for normal Coulomb frames N. Namely, let $X\colon B\to\mathbb{R}^4$ be conformally parametrized such that

$$\mathscr{T}_X[N]=\iint\limits_B \left\{(T_{1,1}^2)^2+(T_{1,2}^2)^2\right\}dudv.$$

Thus Theorem 7.2 gives

$$\mathscr{T}_X[N]\le 2C(\alpha)^2\|S_{1,12}^2\|^2_{C^\alpha(B)}\cdot\iint\limits_B 1\,dudv=2\pi C(\alpha)^2\|S_{1,12}^2\|^2_{C^\alpha(B)}\,.$$

Analogously, Theorem 7.3 yields

$$\mathscr{T}_X[N]\le \pi c(p)^2\|S_{1,12}^2\|^2_{L^p(B)}\,,\quad p>2.$$

We want to reformulate these two results using the invariant scalar curvature $S=W^{-1}S_{1,12}^2$ of the normal bundle.

Theorem 7.4. *Let the conformally parametrized immersion $X\colon B\to\mathbb{R}^4$ together with a normal Coulomb frame N be given. Then there hold*

$$\mathscr{T}_X[N]\le 2\pi C(\alpha)^2\|SW\|^2_{C^\alpha(B)}$$

for all $\alpha\in(0,1)$ with the real constant $C=C(\alpha)$ from Theorem 7.2, as well as

$$\mathscr{T}_X[N]\le \pi c(p)^2\|SW\|^2_{L^p(B)}$$

for all $p\in(2,+\infty]$ with the real constant $c=c(p)$ from Theorem 7.3.

The following lower bound for the total torsion of normal Coulomb frames N is a special case of a general estimate which we will prove in the next chapter.

Theorem 7.5. *Let the conformally parametrized immersion $X\colon B\to\mathbb{R}^4$ together with a normal Coulomb frame N be given. Assume $S_{1,12}^2\not\equiv 0$. Then it holds*

$$\mathscr{T}_X[N]\ge\left(\frac{\|S_{1,12}^2\|^2_{L^2(B)}}{2(1-\rho)^2\|S_{1,12}^2\|^2_{L^2(B_\rho)}}+\frac{\|\nabla S_{1,12}^2\|^2_{L^2(B)}}{\|S_{1,12}^2\|^2_{L^2(B_\rho)}}\right)^{-1}\|S_{1,12}^2\|^2_{L^2(B_\rho)}>0$$

where $\rho=\rho(S_{1,12}^2)\in(0,1)$ is chosen such that

$$\|S_{1,12}^2\|^2_{L^2(B_\rho)}=\iint\limits_{B_\rho(0)}|S_{1,12}^2|^2\,dudv>0.$$

7.9 An example: Holomorphic graphs in $\mathbb{C}^2$

Let us again consider minimal surface graphs

$$X(w) = (w, \Phi(w)), \quad w = u + iv \in B,$$

with a holomorphic function $\Phi = \varphi + i\psi$. Then the Euler unit normal vectors

$$\widehat{N}_1 = \frac{1}{\sqrt{W}}\,(-\varphi_u, -\varphi_v, 1, 0), \quad \widehat{N}_2 = \frac{1}{\sqrt{W}}\,(-\psi_u, -\psi_v, 0, 1) \tag{7.18}$$

form an orthonormal normal frame with the area element

$$W = 1 + |\nabla\varphi|^2 = 1 + |\Phi_w|^2\,.$$

Note here that

$$\Phi_w = \varphi_w + i\psi_w = \frac{1}{2}\left\{\varphi_u - i\varphi_v + i\psi_u + \psi_v\right\} = \varphi_u - i\varphi_v$$

due to $\varphi_u = \psi_v$, $\varphi_v = -\psi_u$. We especially infer

$$\Delta\varphi = \Delta\psi = 0,$$

i.e. X represents a conformally parametrized minimal graph in $\mathbb{R}^4$.

For the torsion coefficients we compute

$$T_{1,1}^2 = \widehat{N}_{1,u}\cdot\widehat{N}_2 \,=\, \frac{1}{W}\,(-\varphi_{uu}\varphi_v + \varphi_{uv}\varphi_u) \,=\, \frac{1}{2W}\frac{\partial}{\partial v}\,(|\nabla\varphi|^2)\,,$$

$$T_{1,2}^2 = \widehat{N}_{1,v}\cdot\widehat{N}_2 \,=\, -\frac{1}{2W}\frac{\partial}{\partial u}\,(|\nabla\varphi|^2)\,.$$

Consequently it holds

$$\operatorname{div}(T_{1,1}^2, T_{1,2}^2) = 0 \quad \text{in } B.$$

In order to check the boundary condition in (7.4) we introduce polar coordinates $u = r\cos\alpha$, $v = r\sin\alpha$. Since $\frac{1}{r}\frac{\partial}{\partial\alpha} = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$ we obtain

$$(T_{1,1}^2, T_{1,2}^2)\cdot\nu = \frac{1}{2W}\left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right)|\nabla\varphi|^2 = \frac{1}{2W}\frac{\partial}{\partial\alpha}\,|\Phi_w|^2 \quad \text{on } \partial B.$$

Proposition 7.4. *Given the conformally parametrized minimal graph $(w, \Phi(w))$ with a holomorphic function $\Phi = \varphi + i\psi$. Then the Euler ONF $\{N_1, N_2\}$ from (7.18) represents a normal Coulomb frame if and only if $|\Phi_w|$ is constant on ∂B.*

In particular, this result applies to graphs $X(w) = (w, w^n)$ for arbitrary $n \in \mathbb{N}$!

7.10 Application to the mean curvature flow in $\mathbb{R}^4$

We consider a family of immersions $X(u,v;\tau)$ evolving by the *mean curvature flow* in direction of the mean curvature vector H as follows

$$X_\tau \equiv \frac{\partial X}{\partial \tau} = -\sum_{\sigma=1}^{2} H_\sigma N_\sigma = -H.$$

We want to compute the evolution of the scalar curvature $S = W^{-1}S_{1,12}^2$ of the normal bundle under this flow. Let $N = (N_1, N_2)$ be an initial ONF.

Lemma 7.7. *The evolution of an unit normal vector N_σ reads*

$$N_{\sigma,\tau} = \sum_{i,j=1}^{2} g^{ij}\left(H_{\sigma,u^i} + \sum_{\varepsilon=1}^{2} H_\varepsilon T_{\varepsilon,i}^\sigma\right) X_{u^j} + \sum_{\varepsilon=1}^{2} T_{\sigma,\tau}^\varepsilon N_\varepsilon \tag{7.19}$$

for $\sigma = 1,2$ with the τ-directed torsion coefficients

$$T_{\sigma,\tau}^\varepsilon = N_{\sigma,\tau} \cdot N_\varepsilon = -N_\sigma \cdot N_{\varepsilon,\tau} = -T_{\varepsilon,\tau}^\sigma .$$

Proof. We make the ansatz

$$\begin{aligned} N_{\sigma,\tau} &= \sum_{i,j=1}^{2} g^{ij}(N_{\sigma,\tau} \cdot X_{u^i})X_{u^j} + \sum_{\varepsilon=1}^{2} (N_{\sigma,\tau} \cdot N_\varepsilon)N_\varepsilon \\ &= \sum_{i,j=1}^{2} g^{ij}(N_{\sigma,\tau} \cdot X_{u^i})X_{u^j} + \sum_{\varepsilon=1}^{2} T_{\sigma,\tau}^\varepsilon N_\varepsilon . \end{aligned}$$

On the other hand, due to the mean curvature flow equation, differentiation of the identity $X_{u^i} \cdot N_\sigma = 0$ w.r.t. τ yields

$$\begin{aligned} N_{\sigma,\tau} \cdot X_{u^i} &= -X_{u^i\tau} \cdot N_\sigma = \left(\sum_{\varepsilon=1}^{2} H_\varepsilon N_\varepsilon\right)_{u^i} \cdot N_\sigma \\ &= (H_{\varepsilon,u^i})N_\varepsilon \cdot N_\sigma + \sum_{\varepsilon=1}^{2} H_\varepsilon T_{\varepsilon,i}^\vartheta N_\vartheta \cdot N_\sigma \end{aligned}$$

proving the statement. □

Thus setting

$$a_1^i := \sum_{j=1}^{2} g^{ij}(H_{1,u^j} + H_2 T_{2,j}^1), \quad a_2^i := \sum_{j=1}^{2} g^{ij}(H_{2,u^j} + H_1 T_{1,j}^2)$$

we find

$$N_{1,\tau} = \sum_{i=1}^{2} a_1^i X_{u^i} + T_{1,\tau}^2 N_2 , \quad N_{2,\tau} = \sum_{i=1}^{2} a_2^i X_{u^i} + T_{2,\tau}^1 N_1 .$$

Theorem 7.6. *The evolution of the scalar curvature S of the normal bundle under the mean curvature flow is given by divergence term*

$$S_\tau = \sum_{i=1}^{2}(-a_1^i L_{2,i2} + a_2^i L_{1,i2})_u + \sum_{i=1}^{2}(a_1^i L_{2,i1} - a_2^i L_{1,i1})_v \,.$$

Proof. We calculate

$$\begin{aligned}
\frac{\partial T_{1,1}^2}{\partial \tau} &= (N_{1,u}\cdot N_2)_\tau = (N_{1,\tau u})\cdot N_2 + N_{1,u}\cdot N_{2,\tau} \\
&= \left(\sum_{i=1}^{2} a_1^i X_{u^i} + T_{1,\tau}^2 N_2\right)_u \cdot N_2 + \dots \\
&\qquad \dots + \left(-\sum_{m,n=1}^{2} L_{1,1m} g^{mn} X_{u^n} + T_{1,1}^2 N_2\right)\cdot\left(\sum_{i=1}^{2} a_2^i X_{u^i} + T_{2,\tau}^1 N_1\right) \\
&= \sum_{i=1}^{2} a_1^i X_{u^i u}\cdot N_2 + \frac{\partial T_{1,\tau}^2}{\partial u} - \sum_{i=1}^{2}\sum_{m,n=1}^{2} a_2^i L_{1,1m} g^{mn} g_{ni} \\
&= \sum_{i=1}^{2} a_1^i L_{2,i1} + \frac{\partial T_{1,\tau}^2}{\partial u} - \sum_{i=1}^{2} a_2^i L_{1,1i}
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{\partial T_{1,2}^2}{\partial \tau} &= (N_{1,v}\cdot N_2)_\tau = (N_{1,\tau v})\cdot N_2 + N_{1,v}\cdot N_{2,\tau} \\
&= \sum_{i=1}^{2} a_1^i X_{u^i v}\cdot N_2 + \frac{\partial T_{1,\tau}^2}{\partial v} - \sum_{i=1}^{2}\sum_{m,n=1}^{2} a_2^i L_{1,2m} g^{mn} g_{ni} \\
&= \sum_{i=1}^{2} a_1^i L_{2,i2} + \frac{\partial T_{1,\tau}^2}{\partial v} - \sum_{i=1}^{2} a_2^i L_{1,2i} \,.
\end{aligned}$$

It follows that

$$\begin{aligned}
S_\tau &= \frac{\partial^2 T_{1,1}^2}{\partial \tau \partial v} - \frac{\partial^2 T_{1,2}^2}{\partial \tau \partial u} \\
&= \sum_{i=1}^{2} \operatorname{div}(-a_1^i L_{2,i2} + a_2^i L_{1,i2}, a_1^i L_{2,i1} - a_2^i L_{1,i1}) - \frac{\partial^2 T_{1,\tau}^2}{\partial u \partial v} + \frac{\partial^2 T_{1,\tau}^2}{\partial v \partial u} \\
&= \sum_{i=1}^{2} \operatorname{div}(-a_1^i L_{2,i2} + a_2^i L_{1,i2}, a_1^i L_{2,i1} - a_2^i L_{1,i1})
\end{aligned}$$

as stated. □

Evolution of the torsion coefficients

We want to refer the reader's attention to an interesting artefact arising with the evolution of the torsion coefficients $T_{\sigma,i}^{\vartheta}$. Consider for this purpose the general case $n \geq 2$ of arbitrary codimensions.

Lemma 7.8. *It holds*

$$N_{\sigma,u^i} \cdot \partial_\tau N_\vartheta = -\sum_{j,k=1}^{2} L_{\sigma,ij} g^{jk} H_{\vartheta,u^k} + \sum_{i,j=1}^{2}\sum_{\omega=1}^{n} H_\omega L_{\sigma,ij} g^{jk} T_{\vartheta,k}^{\omega} - \sum_{\omega=1}^{n} T_{\sigma,i}^{\omega} T_{\omega,\tau}^{\vartheta}$$

for all $i = 1,2$ *and* $\sigma, \vartheta = 1,\ldots,n$.

Proof. We compute

$$\begin{aligned} N_{\sigma,u^i} \cdot \partial_\tau N_\vartheta \;&=\; \sum_{j,k=1}^{2} g^{kj} H_{\vartheta,u^k} X_{u^j} \cdot N_{\sigma,u^i} - \sum_{j,k=1}^{2}\sum_{\omega=1}^{n} g^{kj} T_{\vartheta,k}^{\omega} H_\omega X_{u^j} \cdot N_{\sigma,u^i} \\ &\quad + \sum_{\omega=1}^{n} T_{\vartheta,\tau}^{\omega} N_\omega \cdot N_{\sigma,u^i} \\ &= -\sum_{j,k=1}^{n} g^{kj} H_{\vartheta,u^k} L_{\sigma,ij} + \sum_{j,k=1}^{2}\sum_{\omega=1}^{n} L_{\sigma,ij} g^{kj} H_\omega T_{\vartheta,k}^{\omega} + \sum_{\omega=1}^{n} T_{\vartheta,\tau}^{\omega} T_{\sigma,i}^{\omega} \end{aligned}$$

proving the statement. □

Lemma 7.9. *It holds*

$$\begin{aligned} \partial_\tau N_{\sigma,u^i} \cdot N_\vartheta \;&=\; \sum_{r,s=1}^{2} g^{rs} H_{\sigma,u^r} L_{\vartheta,is} - \sum_{r,s=1}^{2}\sum_{\omega=1}^{n} g^{rs} H_\omega T_{\sigma,r}^{\omega} L_{\vartheta,is} + \partial_{u^i} T_{\sigma,\tau}^{\vartheta} \\ &\quad + \sum_{\omega=1}^{n} T_{\sigma,\tau}^{\omega} T_{\omega,i}^{\vartheta} \end{aligned}$$

for all $i = 1,2$ *and* $\sigma, \vartheta = 1,\ldots,n$.

Proof. For we compute

$$\begin{aligned} \partial_\tau N_{\sigma,u^i} \cdot N_\vartheta \;&=\; \sum_{r,s=1}^{2} g^{rs} H_{\sigma,u^r} X_{u^i u^s} \cdot N_\vartheta - \sum_{r,s=1}^{2}\sum_{\omega=1}^{n} g^{rs} H_\omega T_{\sigma,r}^{\omega} X_{u^i u^s} \cdot N_\vartheta \\ &\quad + \sum_{\omega=1}^{n} \partial_{u^i} T_{\sigma,\tau}^{\omega} \delta_{\omega\vartheta} + \sum_{\omega=1}^{n} T_{\sigma,\tau}^{\omega} N_{\omega,u^i} \cdot N_\vartheta \\ &= \sum_{r,s=1}^{2} g^{rs} H_{\sigma,u^r} L_{\sigma,is} - \sum_{r,s=1}^{2}\sum_{\omega=1}^{n} g^{rs} H_\omega T_{\sigma,r}^{\omega} L_{\vartheta,is} + \partial_{u^i} T_{\sigma,\tau}^{\vartheta} + \sum_{\omega=1}^{n} T_{\sigma,\tau}^{\omega} T_{\omega,i}^{\vartheta} \end{aligned}$$

proving the statement. □

Using these two results we are able to compute the evolution of the torsion coefficients.

Proposition 7.5. *It holds*

$$\partial_\tau T_{\sigma,i}^{\vartheta} = \sum_{j,k=1}^{2} g^{jk}(L_{\vartheta,ik}H_{\sigma,u^i} - L_{\sigma,ij}H_{\vartheta,u^k}) + \sum_{j,k=1}^{2}\sum_{\omega=1}^{n} g^{jk}H_\omega(L_{\sigma,ij}T_{\vartheta,k}^{\omega} - L_{\vartheta,ik}T_{\sigma,j}^{\omega})$$
$$+\partial_{u^i}T_{\sigma,\tau}^{\vartheta} + \sum_{\omega=1}^{n}(T_{\sigma,\tau}^{\omega}T_{\omega,i}^{\vartheta} - T_{\sigma,i}^{\omega}T_{\omega,\tau}^{\vartheta})$$

Proof. This follows from

$$\begin{aligned}
\partial_\tau T_{\sigma,i}^{\vartheta} &= \partial_\tau(N_{\sigma,u^i}\cdot N_\vartheta) = \partial_\tau N_{\sigma,u^i}\cdot N_\vartheta + N_{\sigma,u^i}\cdot\partial_\tau N_\vartheta \\
&= \sum_{r,s=1}^{2} g^{rs}H_{\sigma,u^r}L_{\vartheta,is} - \sum_{r,s=1}^{2}\sum_{\omega=1}^{n} g^{rs}H_\omega T_{\sigma,r}^{\omega}L_{\vartheta,is} + \partial_{u^i}T_{\sigma,\tau}^{\vartheta} + \sum_{\omega=1}^{n} T_{\sigma,\tau}^{\omega}T_{\omega,i}^{\vartheta} \\
&= -\sum_{j,k=1}^{2} L_{\sigma,ij}g^{jk}H_{\vartheta,u^k} + \sum_{j,k=1}^{2}\sum_{\omega=1}^{n} L_{\sigma,ij}g^{jk}H_\omega T_{\vartheta,k}^{\omega} - \sum_{\omega=1}^{n} T_{\sigma,i}^{\omega}T_{\omega,\tau}^{\vartheta} \\
&= \sum_{j,k=1}^{2} g^{jk}(L_{\vartheta,ik}H_{\sigma,u^j} - L_{\sigma,ij}H_{\vartheta,u^k}) + \sum_{j,k=1}^{2}\sum_{\omega=1}^{n} g^{jk}H_\omega(L_{\sigma,ij}T_{\vartheta,k}^{\omega} - L_{\vartheta,ik}T_{\sigma,j}^{\omega}) \\
&\quad + \partial_{u^i}T_{\sigma,\tau}^{\vartheta} + \sum_{\omega=1}^{n}(T_{\sigma,\tau}^{\omega}T_{\omega,i}^{\vartheta} - T_{\sigma,i}^{\omega}T_{\omega,\tau}^{\vartheta})
\end{aligned}$$

as stated. □

Let us now consider the evolution of a minimal surface in $\mathbb{R}^4$ under mean curvature flow with the property $H \equiv 0$ for all times τ. We obtain

$$\partial_\tau T_{1,1}^2 = \partial_u T_{1,\tau}^2\,, \quad \partial_\tau T_{1,2}^2 = \partial_v T_{1,\tau}^2$$

and even though flow does not affect the geometry of the surfaces it actually has consequences of the evolution of the ONF!

Gauge of the time torsion coefficients

The idea to ensure that the whole ONF remains unaffected in this special case just discussed is to introduce an *additional gauge for the time torsion coefficients* $T_{\sigma,\tau}^{\vartheta}$.

This involves various technical difficulties even in case $n = 2$. Hence we want to present an algorithm which at least guarantees *that a normal Coulomb frame actually remains a normal Coulomb frame under the flow.*

For this purpose we first have to establish the Euler-Lagrange equation for normal Coulomb frames in arbitrary parametizations. Using our SO(2)-action

$$\widetilde{N}_1 = \cos\varphi N_1 + \sin\varphi N_2\,, \quad \widetilde{N}_2 = -\sin\varphi N_1 + \cos\varphi N_2$$

we proceed as follows:

$$\begin{aligned}
\mathscr{T}_X[\widetilde{N}] &= \sum_{i,j=1}^{2} \iint_B g^{ij}(T_{1,i}^2 + \varphi_{u^i})(T_{1,j}^2 + \partial_{u^j})W\,dudv \\
&= \mathscr{T}_X[N] + \sum_{i,j=1}^{2} \iint_B g^{ij}(T_{1,i}^2\varphi_{u^j} + T_{1,j}^2\varphi_{u^i})W\,dudv + o(\varphi_u,\varphi_v) \\
&= \mathscr{T}_X[N] + 2\sum_{j=1}^{2} \iint_B (g^{j1}WT_{1,j}^2, g^{j2}WT_{1,j}^2)\cdot(\varphi_u,\varphi_v)\,dudv + o(\varphi_u,\varphi_v) \\
&= \mathscr{T}_X[N] - 2\sum_{i,j=1}^{2} \iint_B \partial_{u^i}(g^{ij}WT_{1,j}^2)\varphi\,dudv - 2\sum_{i,j=1}^{2}\int_{\partial B} g^{ij}WT_{1,j}^2\nu_i\,ds \\
&\quad + o(\varphi_u,\varphi_v)
\end{aligned}$$

with the outer unit normal vector $\nu = (\nu_1,\nu_2)$ at the boundary curve ∂B. Thus we have proved the

Lemma 7.10. *A normal Coulomb frame $N = (N_1,N_2)$ for the immersion $X\colon B \to \mathbb{R}^4$ solves the Neumann boundary value problem*

$$\sum_{i,j=1}^{2} \partial_{u^i}(Wg^{ij}T_{\sigma,j}^{\vartheta}) = 0 \quad \text{in } B, \qquad \sum_{i,j=1}^{2} (Wg^{ij}T_{\sigma,i}^{\vartheta})\cdot\nu_j = 0 \quad \text{in } B.$$

In the special case of a conformally parametrized immersion $X\colon B \to \mathbb{R}^4$ we recover our Euler-Lagrange equation from (7.4), namely

$$\operatorname{div}(T_{1,1}^2, T_{1,2}^2) = 0 \quad \text{in } B, \qquad (T_{1,1}^2, T_{1,2}^2)\cdot\nu = 0 \quad \text{on } \partial B.$$

Now back to our problem: Differentiation of the Euler-Lagrange equation taking account of the identity

$$\partial_\tau(Wg^{ij}) = 0 \quad \text{for } i,j = 1,2$$

since the minimal surface remains fixed under mean curvature flow, we infer

$$\begin{aligned}
0 &= \sum_{i,j=1}^{2} \partial_{u^i}\partial_\tau(Wg^{ij}T_{\sigma,j}^{\vartheta}) = \sum_{i,j=1}^{2} \partial_{u^i}(Wg^{ij}\partial_\tau T_{\sigma,j}^{\vartheta}) \\
&= \sum_{i,j=1}^{2} \partial_{u^i}(Wg^{ij}\partial_{u^j}T_{\sigma,\tau}^{\vartheta}) = W\Delta_{ds^2}T_{\sigma,\tau}^{\vartheta}
\end{aligned}$$

with the invariant Laplace-Beltrami operator

$$\Delta_{ds^2}\psi = \sum_{i,j=1}^{2} \partial_{u^i}(Wg^{ij}\partial_{u^j}\psi).$$

This leads us to the following characterization.

Proposition 7.6. *Let $n=2$ and $H\equiv 0$. Then the mean curvature flow does not affect the Coulomb frame property if and only if*

$$\Delta_{ds^2}T_{\sigma,i}^{\vartheta} = 0 \quad \textit{in } B$$

for all $i=1,2$ and $\sigma,\vartheta=1,2$, and for all times τ.

Parallel mean curvature vector

From chapter 2 we recall the definition of *a mean curvature vector H parallel in the normal bundle,* i.e.

$$H_u^{\perp}\equiv 0, \quad H_v^{\perp}\equiv 0,$$

or equivalently

$$H_{1,u}-T_{1,1}^2H_2=0, \quad H_{1,v}-T_{1,2}^2H_2=0,$$

$$H_{2,u}+T_{1,1}^2H_1=0, \quad H_{2,v}+T_{1,2}^2H_2=0$$

in case $n=2$.

Proposition 7.7. *If the mean curvature H is parallel in the normal bundle for all times then the curvature $S=S_{1,12}^2$ is constant in time, i.e. it holds*

$$\partial_\tau S\equiv 0.$$

Proof. From Proposition 7.5 we infer the representations

$$\begin{aligned}
\partial_\tau T_{1,1}^2 = &\sum_{j,k=1}^{2}\Big\{(L_{2,1j}H_{1,u^k}-L_{1,1j}T_{1,k}^2H_1)-(L_{1,1j}H_{2,u^k}+L_{2,1j}T_{1,k}^2H_2)\Big\}g^{jk}\\
&+\partial_u T_{1,\tau}^2\,,\\
\partial_\tau T_{1,2}^2 = &\sum_{j,k=1}^{2}\Big\{(L_{2,2j}H_{1,u^k}-L_{1,2j}T_{1,k}^2H_1)-(L_{1,2j}H_{2,u^k}+L_{2,2k}T_{1,j}^2H_2)\Big\}\\
&+\partial_v T_{1,\tau}^2\,.
\end{aligned}$$

Now we evaluate the parallelity condition to obtain

$$\begin{aligned}
&\sum_{j,k=1}^{2}\Big\{(L_{2,1j}H_{1,u^k}-L_{1,1j}T_{1,k}^2H_1)-(L_{1,1j}H_{2,u^k}+L_{2,1j}T_{1,k}^2H_2)\Big\}g^{jk}\\
&\quad=\sum_{j,k=1}^{2}\Big\{g^{jk}L_{2,1j}T_{1,k}^2H_2-g^{jk}L_{1,1j}T_{1,k}^2H_1+g^{jk}L_{1,1j}T_{1,k}^2H_1-g^{jk}L_{2,1j}T_{1,k}^2H_2\Big\}\\
&\quad=0
\end{aligned}$$

as well as

$$\begin{aligned}
&\sum_{j,k=1}^{2}\Big\{(L_{2,2j}H_{1,u^k}-L_{1,2j}T_{1,k}^2H_1)-(L_{1,2j}H_{2,u^k}+L_{2,2k}T_{1,j}^2H_2)\Big\}g^{jk}\\
&\quad=\sum_{j,k=1}^{2}\Big\{g^{jk}L_{2,2j}T_{1,k}^2H_2-g^{jk}L_{1,2j}T_{1,k}^2H_1+g^{jk}L_{1,2j}T_{1,k}^2H_1-g^{jk}L_{2,2k}T_{1,j}^2H_2\Big\}\\
&\quad=0.
\end{aligned}$$

This implies

$$\partial_\tau T_{1,1}^2=\partial_u T_{1,\tau}^2\,,\quad \partial_\tau T_{1,2}^2=\partial_v T_{1,\tau}^2\,,$$

and therefore it follows that

$$\partial_\tau S=\partial_\tau(\partial_v T_{1,1}^2-\partial_u T_{1,2}^2)=\partial_v\partial_u T_{1,\tau}^2-\partial_u\partial_v T_{1,\tau}^2=0$$

proving the statement. □

It remains open to identitfy geometric constellations where the mean curvature vector is actually parallel in the normal bundle for all times.

For further discussions on the mean curvature flow for special surface classes we want to refer the reader to Terng [156], Liu and Terng [116], or Smoczyk, Wang and Xin [148]. But a satisying theory seems to fail up to the present.

Chapter 8
Normal Coulomb frames in $\mathbb{R}^{n+2}$

In this chapter we consider surfaces in $\mathbb{R}^{n+2}$ and their orthonormal normal frames critical for the functional of total torsion. We extend the results of the previous chapter, but in this general case now the analysis becomes more intricate for the fact that normal Coulomb frames are solutions of non-linear and inhomogeneous elliptic systems of partial differential equations.

8.1 Problem formulation

We want to generalize our results of the previous chapter to the case of arbitrary codimension $n \geq 2$. We start with computing the Euler-Lagrange equations of the following parameter invariant functional of total torsion

$$\mathscr{T}_X[N] = \frac{1}{2}\sum_{i,j=1}^{2}\sum_{\sigma,\vartheta=1}^{n}\iint\limits_B g^{ij}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}W\,dudv$$

for a normal frame $N = (N_1,\ldots,N_n)$. It will turn out that these equations form a nonlinear system of elliptic partial differential equations with quadratic growth in the gradient.

We derive analytical and geometric properties of critical points, and we will establish existence and regularity results for normal frames in case of vanishing curvature of the normal bundle as well as for normal Coulomb frames in the general situation of non-flat normal bundles.

8.2 The Euler-Lagrange equations

Normal Coulomb frames

Due to do Carmo [22], chapter 3, section 2 we can construct a family $\mathbf{R}(w,\varepsilon)$ of rotations from the Lie group $SO(n)$ for an arbitrary skew-symmetric matrix

$$\mathbf{A}(w) = (a_{\sigma\vartheta}(w))_{\sigma,\vartheta=1,\ldots,n} \in C^\infty(B,\mathrm{so}(n))$$

by means of the geodesic flow in $SO(n)$, with $so(n)$ the associated Lie algebra.

In terms of such a one-parameter family of rotations

$$\mathbf{R}(w,\varepsilon)=\big(R_\sigma^\vartheta(w,\varepsilon)\big)_{\sigma,\vartheta=1,\ldots,n}\in C^\infty(B\times(-\varepsilon_0,+\varepsilon_0),\mathrm{SO}(n)),$$

with sufficiently small $\varepsilon>0$ such that

$$R(w,0)=\mathbb{E}^n\,,\quad \frac{\partial}{\partial\varepsilon}R(w,0)=A(w)\in C^\infty(B,\mathrm{so}(n))$$

holds true with the n-dimensional unit matrix $\mathbb{E}^n$, we consider variations

$$\widetilde{N}=(\widetilde{N}_1,\ldots,\widetilde{N}_n)$$

of a given orthonormal normal frame $N=(N_1,\ldots,N_n)$ by means of

$$\widetilde{N}_\sigma(w,\varepsilon):=\sum_{\vartheta=1}^n r_{\sigma\vartheta}(w,\varepsilon)N_\vartheta(w),\quad \sigma=1,\ldots,n.$$

Such a matrix $\mathbf{A}(w)$ is the essential ingredient for the following definition of the first variation of total functional.

Definition 8.1. An orthonormal normal frame N is called *critical for the functional of total torsion* or a *normal Coulomb frame* if and only if the first variation

$$\delta\mathscr{T}_X[N;A]:=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\mathscr{T}_X[\widetilde{N}]-\mathscr{T}_X[N]\right\}$$

vanishes w.r.t. all skew-symmetric perturbations $\mathbf{A}(w)\in C^\infty(B,\mathrm{so}(n))$.

Computation of the first variation

Now we come to the computation of the first variation of the functional $\mathscr{T}_X[N]$ and determine the according Euler-Lagrange equations.

Proposition 8.1. *The ONF N is a normal Coulomb frame if and only if its torsion coefficients solve the following system of Neumann boundary value problems*

$$\mathrm{div}\,(T_{\sigma,1}^\vartheta,T_{\sigma,2}^\vartheta)=0\quad\textit{in } B,\quad (T_{\sigma,1}^\vartheta,T_{\sigma,2}^\vartheta)\cdot\nu=0\quad \textit{on }\partial B$$

for all $\sigma,\vartheta=1,\ldots,n$, and where ν denotes the outer unit normal vector along the boundary curve ∂B.

Compare it with the single Euler-Lagrange equation in case $n=2$ of two codimensions

$$\mathrm{div}\,(T_{1,1}^2,T_{1,2}^2)=0\quad\text{in } B,\quad (T_{1,1}^2,T_{1,2}^2)\cdot\nu=0\quad\text{on }\partial B.$$

Proof. We consider the one-parameter family of rotations

$$\mathbf{R}(w,\varepsilon) = (r_{\sigma\vartheta}(w,\varepsilon))_{\sigma,\vartheta=1,\ldots,n}$$

as in the introduction. Expanding around $\varepsilon = 0$ yields

$$\mathbf{R}(w,\varepsilon) = \mathbb{E}^n + \varepsilon\mathbf{A}(w) + o(\varepsilon).$$

Now we apply the rotation $\mathbf{R}(w,\varepsilon)$ to the given ONF N. The resulting unit normal vectors $\widetilde{N}_1,\ldots,\widetilde{N}_n$ are then determined by

$$\widetilde{N}_\sigma = \sum_{\vartheta=1}^n r_{\sigma\vartheta}N_\vartheta = \sum_{\vartheta=1}^n \big\{\delta_{\sigma\vartheta} + \varepsilon a_{\sigma\vartheta} + o(\varepsilon)\big\}N_\vartheta = N_\sigma + \varepsilon\sum_{\vartheta=1}^n a_{\sigma\vartheta}N_\vartheta + o(\varepsilon),$$

and for their derivatives we compute

$$\widetilde{N}_{\sigma,u^\ell} = N_{\sigma,u^\ell} + \varepsilon\sum_{\vartheta=1}^n \big(a_{\sigma\vartheta,u^\ell}N_\vartheta + a_{\sigma\vartheta}N_{\vartheta,u^\ell}\big) + o(\varepsilon).$$

Consequently, the new torsion coefficients can be expanded to

$$\begin{aligned}
\widetilde{T}_{\sigma,\ell}^{\omega} &= \widetilde{N}_{\sigma,u^\ell}\cdot\widetilde{N}_\omega \\
&= N_{\sigma,u^\ell}\cdot N_\omega + \varepsilon\sum_{\vartheta=1}^n \big(a_{\sigma\vartheta,u^\ell}N_\vartheta + a_{\sigma\vartheta}N_{\vartheta,u^\ell}\big)\cdot N_\omega \\
&\quad + \varepsilon N_{\sigma,u^\ell}\cdot\sum_{\vartheta=1}^n a_{\omega\vartheta}N_\vartheta + o(\varepsilon) \\
&= T_{\sigma,\ell}^{\omega} + \varepsilon a_{\sigma\omega,u^\ell} + \varepsilon\sum_{\vartheta=1}^n \big\{a_{\sigma\vartheta}T_{\vartheta,\ell}^{\omega} + a_{\omega\vartheta}T_{\sigma,\ell}^{\vartheta}\big\} + o(\varepsilon)
\end{aligned}$$

such that for their squares we infer

$$(\widetilde{T}_{\sigma,\ell}^{\omega})^2 = (T_{\sigma,\ell}^{\omega})^2 + 2\varepsilon\left\{a_{\sigma\omega,u^\ell}T_{\sigma,\ell}^{\omega} + \sum_{\vartheta=1}^n \big(a_{\sigma\vartheta}T_{\vartheta,\ell}^{\omega}T_{\sigma,\ell}^{\omega} + a_{\omega\vartheta}T_{\sigma,\ell}^{\vartheta}T_{\sigma,\ell}^{\omega}\big)\right\} + o(\varepsilon).$$

Before we insert this result into the functional $\mathscr{T}_X[N]$ of total torsion we observe

$$\begin{aligned}
\sum_{\sigma,\omega,\vartheta=1}^n \big\{a_{\sigma\vartheta}T_{\vartheta,\ell}^{\omega}T_{\sigma,\ell}^{\omega} + a_{\omega\vartheta}T_{\sigma,\ell}^{\vartheta}T_{\sigma,\ell}^{\omega}\big\} &= \sum_{\sigma,\omega,\vartheta=1}^n \big\{a_{\sigma\vartheta}T_{\vartheta,\ell}^{\omega}T_{\sigma,\ell}^{\omega} + a_{\sigma\vartheta}T_{\omega,\ell}^{\vartheta}T_{\omega,\ell}^{\sigma}\big\} \\
&= 2\sum_{\sigma,\omega,\vartheta=1}^n a_{\sigma\vartheta}T_{\vartheta,\ell}^{\omega}T_{\sigma,\ell}^{\omega} = 0
\end{aligned}$$

taking the skew-symmetry of $\mathbf{A}(w)$ into account.

Thus the difference between $\mathscr{T}_X[\widetilde{N}]$ and $\mathscr{T}_X[N]$ computes to (notice that it holds $a^{\omega}_{\sigma,u^\ell}T^{\omega}_{\sigma,\ell}=a^{\sigma}_{\omega,u^\ell}T^{\sigma}_{\omega,\ell}$)

$$\begin{aligned}\mathscr{T}_X[\widetilde{N}]-\mathscr{T}_X[N] &= \varepsilon\sum_{\ell=1}^{2}\sum_{\sigma,\omega=1}^{n}\iint\limits_B a_{\sigma\omega,u^\ell}T^{\omega}_{\sigma,\ell}\,dudv+o(\varepsilon)\\ &= 2\varepsilon\sum_{1\le\sigma<\omega\le n}\iint\limits_B\Big\{a_{\sigma\omega,u}T^{\omega}_{\sigma,1}+a_{\sigma\omega,v}T^{\omega}_{\sigma,2}\Big\}\,dudv+o(\varepsilon)\\ &= 2\varepsilon\sum_{1\le\sigma<\omega\le n}\int\limits_{\partial B}a_{\sigma\omega}(T^{\omega}_{\sigma,1},T^{\omega}_{\sigma,2})\cdot\nu\,ds\\ &\quad -2\varepsilon\sum_{1\le\sigma<\omega\le n}^{n}\iint\limits_B a_{\sigma\omega}\,\mathrm{div}\,(T^{\omega}_{\sigma,1},T^{\omega}_{\sigma,2})\,dudv+o(\varepsilon).\end{aligned}$$

But $\mathbf{A}(w)$ was chosen arbitrarily which proves the proposition. □

The integral functions

Interpreting the Euler-Lagrange equations as integrability conditions analogously to the situation considered in section 7.6, Poincare's lemma ensures the existence of integral functions $\tau^{(\sigma\vartheta)}\in C^3(B,\mathbb{R})$ satisfying

$$\nabla\tau^{(\sigma\vartheta)}=\big(-T^{\vartheta}_{\sigma,2},T^{\vartheta}_{\sigma,1}\big)\quad\text{in } B\quad\text{for all }\sigma,\vartheta=1,\ldots,n.$$

Furthermore, due to the Neumann boundary conditions $(T^{\vartheta}_{\sigma,1},T^{\vartheta}_{\sigma,2})\cdot\nu=0$, which imply $\nabla\tau^{(\sigma\vartheta)}\cdot(-v,u)=0$ on ∂B with the unit tangent vector $(-v,u)\perp\nu$ at ∂B, we may again choose $\tau^{(\sigma\vartheta)}$ so that

$$\tau^{(\sigma\vartheta)}=0\quad\text{on }\partial B\quad\text{for all }\sigma,\vartheta=1,\ldots,n.$$

Notice that the matrix $(\tau^{(\sigma\vartheta)})_{\sigma,\vartheta=1,\ldots,n}$ is skew-symmetric.

A nonlinear elliptic system for the integral functions $\tau^{\sigma\vartheta}$

Let us now define the skew-symmetric matrix

$$\delta\tau^{(\sigma\vartheta)}:=\sum_{\omega=1}^{n}\det\Big(\nabla\tau^{(\sigma\omega)},\nabla\tau^{(\omega\vartheta)}\Big),\quad \sigma,\vartheta=1,\ldots n.$$

Our aim now is to establish an elliptic system for $\tau^{(\sigma\vartheta)}$ with quadratic growth in the gradient.

Proposition 8.2. *Let the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ together with normal Coulomb frame N be given. Then the integral functions $\tau^{(\sigma\vartheta)}$ are solutions of the boundary value problems*

$$\Delta\tau^{(\sigma\vartheta)} = -\,\delta\tau^{(\sigma\vartheta)} + S_{\sigma,12}^{\vartheta} \quad \text{in } B, \quad \tau^{(\sigma\vartheta)} = 0 \quad \text{on } \partial B,$$

for $\sigma,\vartheta = 1,\ldots,n$, where $\delta\tau^{(\sigma\vartheta)}$ grows quadratically in the gradient $\nabla\tau^{(\sigma\vartheta)}$.

Proof. Choose any $(\sigma,\vartheta) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$. The representation formula

$$S_{\sigma,12}^{\vartheta} = \partial_v T_{\sigma,1}^{\vartheta} - \partial_u T_{\sigma,2}^{\vartheta} + \sum_{\omega=1}^{n} \Big\{ T_{\sigma,1}^{\omega} T_{\omega,2}^{\vartheta} - T_{\sigma,2}^{\omega} T_{\omega,1}^{\vartheta} \Big\}$$

for the normal curvature tensor together with $\nabla\tau^{(\sigma\vartheta)} = (-T_{\sigma,2}^{\vartheta}, T_{\sigma,1}^{\vartheta})$ yields

$$\begin{aligned}
\Delta\tau^{(\sigma\vartheta)} &= \partial_v T_{\sigma,1}^{\vartheta} - \partial_u T_{\sigma,2}^{\vartheta} \\
&= -\sum_{\omega=1}^{n} \Big\{ T_{\sigma,1}^{\omega} T_{\omega,2}^{\vartheta} - T_{\sigma,2}^{\omega} T_{\omega,1}^{\vartheta} \Big\} + S_{\sigma,12}^{\vartheta} \\
&= \sum_{\omega=1}^{n} \Big\{ \tau_v^{(\sigma\omega)} \tau_u^{(\omega\vartheta)} - \tau_u^{(\sigma\omega)} \tau_v^{(\omega\vartheta)} \Big\} + S_{\sigma,12}^{\vartheta}
\end{aligned}$$

proving the statement. □

8.3 Examples

We want to determine the special form of this nonlinear elliptic system in the special cases $n = 2$ and $n = 3$.

The case $n = 2$

There is only one integral function $\tau^{(12)}$ satisfying

$$\Delta\tau^{(12)} = S_{1,12}^{2} \quad \text{in } B, \quad \tau^{(12)} = 0 \quad \text{on } \partial B.$$

This is exactly the Poisson equation with homogeneous boundary data from section 7.6 with $\tau = \tau^{(12)}$.

The case $n = 3$

If $n = 3$ we have three relations

$$\Delta\tau^{(12)} = \tau_v^{(13)}\tau_u^{(32)} - \tau_u^{(13)}\tau_v^{(32)} + S_{1,12}^2,$$
$$\Delta\tau^{(13)} = \tau_v^{(12)}\tau_u^{(23)} - \tau_u^{(12)}\tau_v^{(23)} + S_{1,12}^3,$$
$$\Delta\tau^{(23)} = \tau_v^{(21)}\tau_u^{(13)} - \tau_u^{(21)}\tau_v^{(13)} + S_{2,12}^3.$$

Recall the curvature vector

$$\mathscr{S} = \frac{1}{W}\,(S_{1,12}^2, S_{1,12}^3, S_{2,12}^3) \in \mathbb{R}^3$$

of the normal bundle from section 3.8. Analogously we define

$$\mathscr{T} := (\tau^{(12)}, \tau^{(13)}, \tau^{(23)}) \in \mathbb{R}^3$$

and infer

$$\Delta\mathscr{T} = \mathscr{T}_u \times \mathscr{T}_v + \mathscr{S}W \quad \text{in } B, \quad \mathscr{T} = 0 \quad \text{on } \partial B \tag{8.1}$$

with the usual vector product $\times$ in $\mathbb{R}^3$.

In other words: *If $n = 3$, then the vector $\mathscr{T}$ solves an inhomogeneous H-surface system with constant mean curvature $H = \frac{1}{2}$ and vanishing boundary data.*

Namely compare it with the mean curvature system

$$\triangle X = 2HWN$$

from chapter 2 with the scalar mean curvature H, the area element W and the unit normal vector N of the surface. If the surface $X\colon B \to \mathbb{R}^5$ would additionally satisfies the conformality relations

$$X_u \cdot X_u = W = X_v \cdot X_v, \quad X_u \cdot X_v = 0 \quad \text{in } B,$$

then X actually represents an immersion with scalar mean curvature H.

We want to point out that $\mathscr{S} \equiv 0$ *is a consequence of* $\mathscr{T} \equiv 0$ *by a result of Wente [165] on systems of the form (8.1), i.e. normal Coulomb frames for surfaces $X\colon B \to \mathbb{R}^5$ with flat normal bundle are free of torsion.*

We consider the general situation of higher codimensions as well as analytical and geometric properties of normal Coulomb frames for surfaces with non-flat normal bundles in the following sections.

8.4 Quadratic growth in the gradient

The Grassmann-type vector $\mathscr{T}$

The previous example gives rise to the definition of the following *vector of Grassmann type*

$$\mathscr{T} := \big(\tau^{(\sigma\vartheta)}\big)_{1\le\sigma<\vartheta\le n} \in \mathbb{R}^N, \quad N := \frac{n}{2}\,(n-1).$$

In our examples this vector $\mathscr{T}$ works as follows:

$$\mathscr{T} = \tau^{(12)} \in \mathbb{R} \qquad \text{for } n=2,$$

$$\mathscr{T} = \big(\tau^{(12)}, \tau^{(13)}, \tau^{(23)}\big) \in \mathbb{R}^3 \qquad \text{for } n=3.$$

Analogously we define

$$\delta\mathscr{T} := \big(\delta\tau^{(\sigma\vartheta)}\big)_{1\le\sigma<\vartheta\le n} \in \mathbb{R}^N.$$

Then Proposition 8.2 can be written in the following succint form

$$\Delta\mathscr{T} = -\delta\mathscr{T} + \mathscr{S}W \quad \text{in } B, \quad \mathscr{T} = 0 \quad \text{on } \partial B.$$

From the definition of $\delta\mathscr{G}$ we next obtain

$$|\Delta\mathscr{T}| \le c\,|\nabla\mathscr{T}|^2 + |\mathscr{S}W| \quad \text{in } B$$

with some real constant $c>0$. The exact knowledge of this constant will become important later.

A nonlinear system with quadratic growth for $\mathscr{T}$

Proposition 8.3. *Let the conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$ together with a normal Coulomb frame N be given. Then it holds*

$$|\Delta\mathscr{T}| \le \frac{\sqrt{n-2}}{2}\,|\nabla\mathscr{T}|^2 + |\mathscr{S}W| \quad \text{in } B.$$

Proof. We already know

$$|\Delta\mathscr{T}| \le |\delta\mathscr{T}| + |\mathscr{S}W| \quad \text{in } B.$$

Thus it remains to estimate $|\delta\mathscr{T}|$ appropriately. We begin with computing

$$|\delta\mathscr{T}|^2 = \sum_{1\le\sigma<\vartheta\le n}\left\{\sum_{\omega=1}^{n}\det\left(\nabla\tau^{(\sigma\omega)},\nabla\tau^{(\omega\vartheta)}\right)\right\}^2$$
$$\le (n-2)\sum_{1\le\sigma<\vartheta\le n}\left\{\sum_{\omega=1}^{n}\det\left(\nabla\tau^{(\sigma\omega)},\nabla\tau^{(\omega\vartheta)}\right)^2\right\}.$$

Note that only derivatives of elements of $\mathscr{T}$ appear on the right hand side, say $\mathscr{R}$, of $|\delta\mathscr{T}|^2$. Moreover this right hand side can be estimated by $|\mathscr{T}_u\wedge\mathscr{T}_v|^2$ since $\mathscr{T}_u\wedge\mathscr{T}_v$ has actually more elements than $\mathscr{R}$.[1] Thus using Lagrange's identity

$$|X\wedge Y|^2 = |X|^2|Y|^2-(X\cdot Y)^2 \le |X|^2|Y|^2$$

for two vectors X and Y we can estimate as follows

$$|\delta\mathscr{T}|^2 \le (n-2)|\mathscr{T}_u\wedge\mathscr{T}_v|^2 \le (n-2)|\mathscr{T}_u|^2|\mathscr{T}_v|^2$$
$$\le \left\{\frac{\sqrt{n-2}}{2}\left(|\mathscr{T}_u|^2+|\mathscr{T}_v|^2\right)\right\}^2.$$

This proves the statement. □

8.5 Torsion free normal frames

The case $n=3$

As already mentioned above, in Wente [165] we find an interesting uniqueness result for solutions of the homogeneous system

$$\Delta\mathscr{T} = \mathscr{T}_u\times\mathscr{T}_v \quad \text{in } B$$

with vanishing boundary values, corresponding to the flat normal bundle situation $\mathscr{S}\equiv 0$ in case $n=3$. Wente's results particularly states

Proposition 8.4. *(Wente [165])*
The only solution of the elliptic system

$$\Delta\mathscr{T} = \mathscr{T}_u\times\mathscr{T}_v \quad \text{in } B, \quad \mathscr{T}=0 \quad \text{on } \partial B,$$

is the trivial solution $\mathscr{T}\equiv 0$.

---

[1] In particular, elements of the form $\det(\nabla\tau^{(\sigma\omega)},\nabla\tau^{(\omega'\vartheta)})^2$ appear in $|\mathscr{T}_u\wedge\mathscr{T}_v|^2$, but they do not appear in $\mathscr{R}$.

Below we will give a new proof of Wente's result, independent of the codimension, which follows from asymptotic expansions of solutions $\mathscr{T}$ in the interior B and on the boundary ∂B.

For the present we can confirm

Corollary 8.1. *Suppose that the immersion $X\colon B \to \mathbb{R}^5$ admits a normal Coulomb frame. Then this frame is free of torsion if and only if the curvature vector $\mathscr{S}$ vanishes identically.*

An auxiliary function

For a complete treatment of the general case $n > 3$ we need some further preparations. Let us start with the

Lemma 8.1. *Let $\mathscr{S} \equiv 0$. Then the function*

$$\Phi(w) = \mathscr{T}_w(w) \cdot \mathscr{T}_w(w) = \sum_{1\le\sigma<\vartheta\le n} \tau_w^{(\sigma\vartheta)} \tau_w^{(\sigma\vartheta)} ,$$

using the complex notation $\phi_w = \frac{1}{2}(\phi_u + i\phi_v)$, vanishes identically in B.

Proof. We will prove that Φ solves the boundary value problem

$$\Phi_{\overline{w}} = 0 \quad \text{in } B, \quad \operatorname{Im}(w^2\Phi) = 0 \quad \text{on } \partial B.$$

Then the analytic function

$$\Psi(w) := w^2\Phi(w)$$

has vanishing imaginary part, and the Cauchy-Riemann equations imply $\Psi(w) \equiv c$ with some $c \in \mathbb{R}$. The assertion follows from $\Psi(0) = 0$.

1. In order to deduce the stated boundary condition, recall that $\tau^{(\sigma\vartheta)} = 0$ on ∂B. Thus all tangential derivatives vanish identically because

$$-v\tau_u^{(\sigma\vartheta)} + u\tau_v^{(\sigma\vartheta)} = -2\operatorname{Im}(w\tau_w^{(\sigma\vartheta)}) = 0 \quad \text{on } \partial B$$

for all $\sigma, \vartheta = 1, \ldots, n$. The boundary condition follows from

$$\begin{aligned} \operatorname{Im}(w^2\Phi) &= \operatorname{Im}\Big(w^2\, \mathscr{T}_w \cdot \mathscr{T}_w\Big) = \operatorname{Im}\Big\{w^2 \sum_{1\le\sigma<\vartheta\le n} \tau_w^{(\sigma\vartheta)} \tau_w^{(\sigma\vartheta)}\Big\} \\ &= \sum_{1\le\sigma<\vartheta\le n} \operatorname{Im}\Big\{\big(w\tau_w^{(\sigma\vartheta)}\big)\big(w\tau_w^{(\sigma\vartheta)}\big)\Big\} \\ &= 2\sum_{1\le\sigma<\vartheta\le n} \operatorname{Re}\big(w\tau_w^{(\sigma\vartheta)}\big) \operatorname{Im}\big(w\tau_w^{(\sigma\vartheta)}\big) = 0 \end{aligned}$$

on the boundary ∂B.

2. Now we show the analyticity of Φ using

$$\Delta\tau^{(\sigma\vartheta)} = 4\tau^{(\sigma\vartheta)}_{w\overline{w}} = -\delta\tau^{(\sigma\vartheta)}.$$

Interchanging indices cyclically yields

$$\begin{aligned}
2\Phi_{\overline{w}} = 4\,\mathscr{T}_w\cdot\mathscr{T}_{w\overline{w}} = 4\sum_{1\le\sigma<\vartheta\le n}\tau^{(\sigma\vartheta)}_w\tau^{(\sigma\vartheta)}_{w\overline{w}} = \frac{1}{2}\sum_{\sigma,\vartheta=1}^{n}\tau^{(\sigma\vartheta)}_w\Delta\tau^{(\sigma\vartheta)}\\
= \frac{1}{4}\sum_{\sigma,\vartheta,\omega=1}^{n}\left\{\tau^{(\sigma\omega)}_v\tau^{(\omega\vartheta)}_u\tau^{(\sigma\vartheta)}_u - \tau^{(\sigma\omega)}_u\tau^{(\omega\vartheta)}_v\tau^{(\sigma\vartheta)}_u\right\}\\
-\frac{i}{4}\sum_{\sigma,\vartheta,\omega=1}^{n}\left\{\tau^{(\sigma\omega)}_v\tau^{(\omega\vartheta)}_u\tau^{(\sigma\vartheta)}_v - \tau^{(\sigma\omega)}_u\tau^{(\omega\vartheta)}_v\tau^{(\sigma\vartheta)}_v\right\}\\
= \frac{1}{4}\sum_{\sigma,\vartheta,\omega=1}^{n}\left\{\tau^{(\omega\vartheta)}_v\tau^{(\vartheta\sigma)}_u\tau^{(\omega\sigma)}_u - \tau^{(\sigma\omega)}_u\tau^{(\omega\vartheta)}_v\tau^{(\sigma\vartheta)}_u\right\}\\
-\frac{i}{4}\sum_{\sigma,\vartheta,\omega=1}^{n}\left\{\tau^{(\vartheta\sigma)}_v\tau^{(\sigma\omega)}_u\tau^{(\vartheta\omega)}_v - \tau^{(\sigma\omega)}_u\tau^{(\omega\vartheta)}_v\tau^{(\sigma\vartheta)}_v\right\}
\end{aligned}$$

which shows $\Phi_{\overline{w}} = 0$. The proof is complete. □

The case $n \ge 3$

The main result of this section is the

Theorem 8.1. *Suppose that the immersion $X\colon B\longrightarrow\mathbb{R}^{n+2}$ admits a normal Coulomb frame N. Then this frame is free of torsion if and only if the curvature vector S of its normal bundle vanishes identically.*

Proof. We introduce conformal parameters $(u,v)\in B$. Furthermore, let N be a normal Coulomb frame. If N is free of torsion then the curvature vector $\mathscr{S}$ vanishes identically. So assume conversely $\mathscr{S}\equiv 0$ and let us show that N is free of torsion. Consider for this aim the Grassmann-type vector $\mathscr{T}$ from above. Since

$$\mathscr{T}_w\cdot\mathscr{T}_w \equiv 0 \quad \text{in } B$$

holds true by the previous lemma, we find

$$|\mathscr{T}_u| = |\mathscr{T}_v|, \quad \mathscr{T}_u\cdot\mathscr{T}_v = 0 \quad \text{in } B.$$

Thus $\mathscr{T}$ is a conformally parametrized solution of

$$\Delta\mathscr{T} = -\delta\mathscr{T} \quad \text{in } B, \quad \mathscr{T} = 0 \quad \text{on } \partial B.$$

According to the quadratic growth condition $|\delta\mathscr{T}| \le c|\nabla\mathscr{T}|^2$ (see paragraph 8.4), the arguments in Heinz [81] apply[2]: Assume $\mathscr{T} \not\equiv \text{const}$ in B. Then the asymptotic expansion stated in Satz of Heinz [81] implies that boundary branch points $w_0 \in \partial B$ with the property

$$\mathscr{T}_u(w_0) = \mathscr{T}_v(w_0) = 0$$

are isolated. But this contradicts our boundary condition $\mathscr{T}|_{\partial B} = 0$! We thus infer

$$\mathscr{T} \equiv \text{const} = 0$$

which implies $\tau^{(\sigma\vartheta)} \equiv 0$ for all $\sigma, \vartheta = 1, \ldots, n$, and the normal Coulomb frame is free of torsion. The theorem is proved. □

8.6 Non-flat normal bundles

We establish upper bounds for the torsion coefficients of normal Coulomb frames.

An upper bound for $\mathscr{T}$ via Wente's L^∞-estimate

Our first result is

Proposition 8.5. *Let N be a normal Coulomb frame for the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$. Then the Grassmann-type vector $\mathscr{T}$ satisfies*

$$\begin{aligned}\|\mathscr{T}\|_{L^\infty(B)} &= \sup_{w\in B} \sqrt{\sum_{1\le\sigma<\vartheta\le n} |\tau^{(\sigma\vartheta)}(w)|^2}\\ &\le \frac{n-2}{2\pi}\,\|\nabla\mathscr{T}\|^2_{L^2(B)} + \frac{n(n-1)}{8}\,\|\mathscr{S}W\|_{L^\infty(B)}\end{aligned}$$

with the Lebesgue norms $L^2(B)$ and $L^\infty(B)$.

Proof. 1. For $1 \le \sigma < \vartheta \le n$, $\omega \in \{1,\ldots,n\}$ with $\omega \notin \{\sigma,\vartheta\}$, and given integral functions $\tau^{(\sigma\vartheta)}$ we define the functions $y^{(\sigma\vartheta\omega)}$ as the unique solutions of

$$\Delta y^{(\sigma\vartheta\omega)} = -\det\left(\nabla\tau^{(\sigma\omega)}, \nabla\tau^{(\omega\vartheta)}\right) \quad \text{in } B, \quad y^{(\sigma\vartheta\omega)} = 0 \quad \text{on } \partial B.$$

---

[2] Let $X\colon \in C^2(B,R^n)$ solve the system $\triangle X = Hf(X,X_u,X_v)$ together with $X_u^2 = X_v^2$, $X_u \cdot X_v = 0$, where $|f(x,p,q)| \le \mu(|x|)(|p|^2+|q|^2)$. Then if $X_u(w_0) = 0$ for some $w_0 \in \partial B$, but $X_u \not\equiv 0$, the asymptotic expansion $X_w(w_0) = a(w-w_0)^\ell - o(|w-w_0|^\ell)$ holds for $w \to w_0$, where $a \in \mathbb{C}$ with $a_1^2 + \ldots a_n^2 = 0$, and $\ell \in \mathbb{N}\setminus\{0\}$.

Wente's L^∞-estimate fom Wente [166] together with Topping [159] then yields the *optimal inequalities*[3]

$$\|y^{(\sigma\vartheta\omega)}\|_{L^\infty(B)} \le \frac{1}{4\pi}\Big(\|\nabla\tau^{(\sigma\omega)}\|^2_{L^2(B)} + \|\nabla\tau^{(\omega\vartheta)}\|^2_{L^2(B)}\Big)$$

for $1 \le \sigma < \vartheta \le n$, $\omega \notin \{\sigma,\vartheta\}$. In addition we introduce the Grassmann-type vector $\mathscr{Z} = (z^{(\sigma\vartheta)})_{1\le\sigma<\vartheta\le n}$ as the unique solution of

$$\Delta\mathscr{Z} = \mathscr{S}W \quad \text{in } B, \quad \mathscr{Z} = 0 \quad \text{on } \partial B.$$

Introduce new indices to write

$$\mathscr{Z} = (Z_1,\ldots,Z_N), \quad \mathscr{S}W = (S_1,\ldots,S_N)$$

for the moment. We use Poisson's representation formula to estimate as follows

$$\begin{aligned}
|\mathscr{Z}(w)| \le \sum_{i=1}^N |Z_i| &= \sum_{i=1}^N \left| \iint_B \phi(\zeta;w)S_i(\zeta)\,d\xi\,d\eta \right| \\
&\le \sum_{i=1}^N \iint_B |\phi(\zeta;w)||S_i(\zeta)|\,d\xi\,d\eta \\
&\le \sqrt{N}\iint_B |\phi(\zeta;w)|\sqrt{\sum_{i=1}^N |S_i(\zeta)|^2}\,d\xi\,d\eta \\
&= \sqrt{N}\iint_B |\phi(\zeta;w)||\mathscr{S}(\zeta)W(\zeta)|\,d\xi\,d\eta
\end{aligned}$$

with the Green function $\phi(\zeta;w)$ for the Laplacian Δ in B; $\zeta = (\xi,\eta)$. From section 7.6 we already know

$$\iint_B |\phi(\zeta;w)|\,d\xi\,d\eta = \frac{1-|w|^2}{4} \le \frac{1}{4}$$

which enables us to continue with estimating $|\mathscr{Z}(w)|$ to get

$$\|\mathscr{Z}\|_{L^\infty(B)} \le \sqrt{N}\|\mathscr{S}W\|_{L^\infty(B)}\iint_B |\phi(\zeta;w)|\,d\xi\,d\eta \le \frac{\sqrt{N}}{4}\|\mathscr{S}W\|_{L^\infty(B)}.$$

---

[3] In 1980 H. Wente proved: Let $\Phi \in C^0(B)\cap H^{1,2}(B)$ be a solution of $\Delta\Phi = -(f_u g_v - f_v g_u)$ in B, $\Phi = 0$ on $\partial\Omega$, with $f,g \in H^{1,2}(B)$, then $\|\Phi\|_{L^\infty(B)} + \|\nabla\Phi\|_{L^2(B)} \le C\|\nabla f\|_{L^2(B)}\|\nabla g\|_{L^2(B)}$. Following Topping [159] we may set $\frac{C}{2} = \frac{1}{4\pi}$ after applying Hölder's inequality.

2. Now recall that

$$\Delta\tau^{(\sigma\vartheta)} = -\sum_{\omega=1}^{n} \det(\nabla\tau^{(\sigma\omega)}, \nabla\tau^{(\omega\vartheta)}) + S_{\sigma,12}^{\vartheta} W = \sum_{\omega=1}^{n} \Delta y^{(\sigma\vartheta\omega)} + \Delta z^{(\sigma\vartheta)}.$$

Taking account of the unique solvability of the above introduced Dirichlet problems with vanishing boundary data, the maximum principle yields

$$\tau^{(\sigma\vartheta)} = \sum_{\omega\notin\{\sigma,\vartheta\}} y^{(\sigma\vartheta\omega)} + z^{(\sigma\vartheta)}, \quad 1 \le \sigma < \vartheta \le n.$$

Now we collect all the estimates obtained and get (rearrange the summations and redefine some indices)

$$\begin{aligned}
\|\mathscr{T}\|_{L^\infty(B)} &\le \sum_{1\le\sigma<\vartheta\le n}\sum_{\omega\notin\{\sigma,\vartheta\}} \|y^{(\sigma\vartheta\omega)}\|_{L^\infty(B)} + \sum_{1\le\sigma<\vartheta\le n} \|z^{(\sigma\vartheta)}\|_{L^\infty(B)} \\
&\le \sum_{1\le\sigma<\vartheta\le n}\sum_{\omega\notin\{\sigma,\vartheta\}} \|y^{(\sigma\vartheta\omega)}\|_{L^\infty(B)} + \sqrt{N}\,\|\mathscr{Z}\|_{L^\infty(B)} \\
&\le \frac{1}{4\pi} \sum_{1\le\sigma<\vartheta\le n}\sum_{\omega\notin\{\sigma,\vartheta\}} \Big(\|\nabla\tau^{(\sigma\omega)}\|_{L^2(B)}^2 + \|\nabla\tau^{(\omega\vartheta)}\|_{L^2(B)}^2\Big) \\
&\quad + \frac{N}{4}\,\|\mathscr{S}W\|_{L^\infty(B)} \\
&= \frac{1}{4\pi}\Bigg\{ \sum_{1\le\omega<\sigma<\vartheta\le n} \Big(\|\nabla\tau^{(\omega\sigma)}\|_{L^2(B)}^2 + \|\nabla\tau^{(\omega\vartheta)}\|_{L^2(B)}^2\Big) \\
&\qquad + \sum_{1\le\sigma<\omega<\vartheta\le n} \Big(\|\nabla\tau^{(\sigma\omega)}\|_{L^2(B)}^2 + \|\nabla\tau^{(\omega\vartheta)}\|_{L^2(B)}^2\Big) \\
&\qquad + \sum_{1\le\sigma<\vartheta<\omega\le n} \Big(\|\nabla\tau^{(\sigma\omega)}\|_{L^2(B)}^2 + \|\nabla\tau^{(\vartheta\omega)}\|_{L^2(B)}^2\Big)\Bigg\} \\
&\quad + \frac{N}{4}\,\|\mathscr{S}W\|_{L^\infty(B)} \\
&= \ldots
\end{aligned}$$

$$\begin{aligned}
\ldots = \frac{1}{4\pi}\Bigg\{ & \sum_{1\le\sigma<\vartheta<\omega\le n} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} + \sum_{1\le\sigma<\omega<\vartheta\le n} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} \\
& + \sum_{1\le\sigma<\vartheta<\omega\le n} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} + \sum_{1\le\omega<\sigma<\vartheta\le n} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} \\
& + \sum_{1\le\sigma<\omega<\vartheta\le n} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} + \sum_{1\le\omega<\sigma<\vartheta\le n} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} \Bigg\} \\
& + \frac{N}{4}\,\|\mathscr{S}W\|_{L^\infty(B)} \\
= \frac{1}{2\pi} & \sum_{1\le\sigma<\vartheta\le n}\sum_{\omega\notin\{\sigma,\vartheta\}} \|\nabla\tau^{(\sigma\vartheta)}\|^2_{L^2(B)} + \frac{N}{4}\,\|\mathscr{S}W\|_{L^\infty(B)} \\
= \frac{n-2}{2\pi} & \,\|\nabla\mathscr{T}\|^2_{L^2(B)} + \frac{1}{4}\,\frac{n(n-1)}{2}\,\|\mathscr{S}W\|_{L^\infty(B)}\,.
\end{aligned}$$

This proves the statement. □

An upper bound for $\mathscr{T}$ via Poincaré's inequality

We want to present an alternative way to establish an upper bound for the Grassmann-type vector $\mathscr{T}$. In particular, we show

$$|z^{(\sigma\vartheta)}(w)| \le \sqrt{\frac{2}{\pi}}\,\|S^{\vartheta}_{\sigma,12}W\|_{L^2(B)} \quad \text{in } B \quad \text{for all } 1\le\sigma<\vartheta\le n$$

for the vector $\mathscr{Z} = (z^{(12)}, z^{(13)}, \ldots)$ from the previous paragraph. This would lead us to

$$\begin{aligned}
\|\mathscr{Z}\|_{L^\infty(B)} &= \sup_B \sqrt{\sum_{1\le\sigma<\vartheta\le n} |z^{(\sigma\vartheta)}(w)|^2} \le \sqrt{\frac{2}{\pi}}\sqrt{\sum_{1\le\sigma<\vartheta\le n} \|S^{\vartheta}_{\sigma,12}W\|^2_{L^2(B)}} \\
&= \sqrt{\frac{2}{\pi}}\,\|\mathscr{S}W\|_{L^2(B)} \le \sqrt{2}\,\|\mathscr{S}W\|_{L^\infty(B)}
\end{aligned}$$

which finally gives a smaller upper bound for $\|\mathscr{T}\|_{L^\infty(B)}$ at least for codimensions $n = 2,3$.

In order to prove the stated inequality we start with the Poisson representation formula

$$z^{(\sigma\vartheta)} = \iint_B \phi(\zeta;w)S^{\vartheta}_{\sigma,12}(\zeta)W(\zeta)\,d\xi\,d\eta, \quad z^{(\sigma\vartheta)} = 0 \quad \text{on } \partial B.$$

Applying the Hölder and the Poincaré inequality gives

$$\begin{aligned} |z^{(\sigma\vartheta)}(w)| &\le \|\phi(\cdot\,;w)\|_{L^2(B)}\|S^{\vartheta}_{\sigma,12}W\|_{L^2(B)} \\ &\le \frac{1}{2\sqrt{\pi}}\,\|\nabla_{\zeta}\phi(\cdot\,;w)\|_{L^1(B)}\|S^{\vartheta}_{\sigma,12}W\|_{L^2(B)}\,. \end{aligned} \tag{8.2}$$

For the optimal constant $\frac{1}{2\sqrt{\pi}}$ in the Sobolev inequality we refer to Gilbarg and Trudinger [71], paragraph 7.7 and the references therein. Furthermore

$$\phi = \phi(\zeta;w) := \frac{1}{2\pi}\log\Big|\frac{\zeta - w}{1-\overline{w}\zeta}\Big|$$

denotes again Green's function for the Laplace operator Δ in B, and it satisfies $\phi(\cdot\,;w) \in H^1_0(B)$ as well as

$$2\phi_{\zeta}(\zeta;w) \equiv \phi_{\xi}(\zeta;w) - i\phi_{\eta}(\zeta;w) = \frac{1}{2\pi}\,\overline{\left(\frac{\zeta-w}{|\zeta-w|^2} + w\frac{1-\overline{w}\zeta}{|1-\overline{w}\zeta|^2}\right)}, \quad w \ne \zeta.$$

A straightforward calculation shows

$$|\nabla_{\zeta}\phi(\zeta;w)| \equiv 2|\phi_{\zeta}(\zeta;w)| = \frac{1}{2\pi}\,\frac{1-|w|^2}{|\zeta-w|\,|1-\overline{w}\zeta|} \le \frac{1}{2\pi}\,\frac{1+|w|}{|\zeta-w|} \le \frac{1}{\pi}\,\frac{1}{|\zeta-w|}\,.$$

for $\zeta \ne w$. And since the right hand side in the inequality

$$\begin{aligned} \iint\limits_{B} |\nabla_{\zeta}\phi(\zeta;w)|\,d\xi\,d\eta &\le \frac{1}{\pi}\iint\limits_{B_{\delta}(w)} \frac{1}{|\zeta-w|}\,d\xi\,d\eta + \frac{1}{\pi}\iint\limits_{B\setminus B_{\delta}(w)} \frac{1}{|\zeta-w|}\,d\xi\,d\eta \\ &\le 2\delta + \frac{1}{\delta} \end{aligned}$$

becomes minimal for $\delta = \frac{1}{\sqrt{2}}$, we arrive at

$$|z^{(\sigma\vartheta)}(w)| \le \frac{1}{2\sqrt{\pi}}\left(2\cdot\frac{1}{\sqrt{2}} + \sqrt{2}\right)\|S^{\vartheta}_{\sigma,12}W\|_{L^2(B)} = \frac{\sqrt{2}}{\sqrt{\pi}}\,\|S^{\vartheta}_{\sigma,12}W\|_{L^2(B)}$$

proving the stated inequality. Rearranging gives

$$|\mathscr{Z}(w)| \le \frac{\sqrt{2}}{\sqrt{\pi}}\cdot\sqrt{\pi}\cdot\|S^{\vartheta}_{\sigma,12}W\|_{L^{\infty}(B)} \le \sqrt{2}\,\|\mathscr{S}W\|_{L^{\infty}(B)}\,.$$

Thus we have the following proposition.

Proposition 8.6. *Let N be a normal Coulomb frame for the conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$. Then the Grassmann-type vector $\mathscr{T}$ satisfies*

$$\|\mathscr{T}\|_{L^\infty(B)}\le\frac{n-2}{2\pi}\|\nabla\mathscr{T}\|^2_{L^2(B)}+\sqrt{2}\,\|\mathscr{S}W\|_{L^\infty(B)}\,.$$

This provides us a better estimate at least for small n.

An estimate for the torsion coefficients

We are now in the position to prove our main result of this section.

Theorem 8.2. *Let N be a normal Coulomb frame for the conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$ with total torsion $\mathscr{T}_X[N]$ and given $\|\mathscr{S}W\|_{L^\infty(B)}$. Assume that the smallness condition*

$$\frac{\sqrt{n-2}}{2}\left(\frac{n-2}{4\pi}\,\mathscr{T}_X[N]+C(n)\|\mathscr{S}W\|_{L^\infty(B)}\right)<1$$

is satisfied with $C(n):=\min\big\{\frac{n(n-1)}{8},\sqrt{2}\big\}$. Then the torsion coefficients of N can be estimated by

$$\|T_{\sigma,i}^{\vartheta}\|_{L^\infty(B)}\le c,\quad i=1,2,\ 1\le\sigma<\vartheta\le n,$$

with a nonnegative constant $c=c(n,\|\mathscr{S}W\|_{L^\infty(B)},\mathscr{T}_X[N])<+\infty$.

Proof. We have the following elliptic system

$$|\Delta\mathscr{T}|\le\sqrt{\frac{n(n-1)}{2}}\,|\nabla\mathscr{T}|^2+|\mathscr{S}W|\quad\text{in }B,\quad \mathscr{T}=0\quad\text{on }\partial B,$$

$$\|\mathscr{T}\|_{L^\infty(B)}\le\frac{n-2}{2\pi}\,\|\nabla\mathscr{T}\|^2_{L^2(B)}+C(n)\|\mathscr{S}W\|_{L^\infty(B)}\le M\in[0,+\infty)\,.$$

The smallness condition ensures that we can apply Heinz's global gradient estimate Theorem 1 in Sauvigny [143], chapter XII, § 3, obtaining $\|\nabla\mathscr{T}\|_\infty\le c$.[4] This in turn yields the desired estimate. □

It remains open to prove global pointwise estimates for the torsion coefficients without the assumed smallness condition. In particular, we would like to get rid of the a priori knowledge of $\mathscr{T}_X[N]$.

[4] This theorem states: Let $X\in C^2(B,\mathbb{R}^n)$ be a solution of the elliptic system $|\Delta X|\le a|\nabla X|^2+b$ in B with $X=0$ on ∂B and $\|X\|_{L^\infty(B)}\le M$. Assume $aM<1$. Then there is a constant $c=c(a,b,M,\alpha)$ such that $\|X\|_{C^{1+\alpha}(B)}\le c(a,b,M,\alpha)$.

8.7 Bounds for the total torsion

Upper bounds

From the torsion estimates above we can immediately infer various upper bounds for the functional of total torsion.

For example, let us focus on *small solutions* $\mathscr{T}$: Namely, multiply

$$\Delta\mathscr{T} = -\delta\mathscr{T} + \mathscr{S}W$$

by $\mathscr{T}$ and integrate by parts yields to get

Proposition 8.7. *For small solutions* $\|\mathscr{T}\|_{L^\infty(B)} \le \frac{2}{\sqrt{n-2}}$ *it holds*

$$\mathscr{T}_X[N] = 2\|\nabla\mathscr{T}\|^2_{L^2(B)} \le \frac{4\|\mathscr{T}\|_{L^\infty(B)}\|\mathscr{S}W\|_{L^1(B)}}{2-\sqrt{n-2}\|\mathscr{T}\|_{L^\infty(B)}}\,.$$

The reader is refered to Sauvigny [143] for the construction of such small solutions of nonlinear elliptic systems. Let us emphasize that the case $n=2$ is much easier to handle: The classical maximum principle controls $\|\mathscr{T}\|_{L^\infty(B)}$ in terms of $\|\mathscr{S}W\|_{L^\infty(B)}$, and no smallness condition is needed to bound the functional of total torsion.

A lower bound

Finally we complete this section with establishing a lower bound for the functional of total torsion.

Proposition 8.8. *Let N a normal Coulomb frame for the conformally parametrized immersion $X\colon B\to\mathbb{R}^{n+2}$. Assume that the curvature vector $\mathscr{S}$ of its normal bundle satisfies $\mathscr{S}\not\equiv 0$ as well as*

$$\|\nabla(\mathscr{S}W)\|_{L^2(B)} > 0.$$

Then it holds

$$\mathscr{T}_X[N] \ge \frac{1}{2}\left(\sqrt{n-2}\,\|\mathscr{S}W\|_{L^\infty(B)} + \frac{\|\mathscr{S}W\|^2_{L^2(B)}}{(1-\rho)^2\|\mathscr{S}W\|^2_{L^2(B_\rho)}} + \ldots\right.$$
$$\left.\ldots + \frac{2\|\nabla(\mathscr{S}W)\|^2_{L^2(B)}}{\|\mathscr{S}W\|^2_{L^2(B_\rho)}}\right)^{-1}\|\mathscr{S}W\|^2_{L^2(B_\rho)} > 0$$

with the radius $\rho=\rho(\mathscr{S})\in(0,1)$ constructed in the proof given below, and the setting $B_\rho=\{(u,v)\in\mathbb{R}^2\, u^2+v^2\le\rho\}$.

Proof. 1. Because of $\mathscr{S} \neq 0$ there exists (a first) $\rho = \rho(\mathscr{S}) \in (0,1)$ such that

$$\|\mathscr{S}W\|_{L^2(B_\rho)} = \left(\iint\limits_{B_\rho(0)} |\mathscr{S}W|^2\,dudv \right)^{\frac{1}{2}} > 0,$$

and this is already our radius ρ from the theorem. Now we choose a test function $\eta \in C^0(B,\mathbb{R}) \cap H_0^{1,2}(B,\mathbb{R})$ such that

$$\eta \in [0,1] \quad \text{in } B, \quad \eta = 1 \quad \text{in } B_\rho\,, \quad |\nabla\eta| \le \frac{1}{1-\rho} \quad \text{in } B.$$

Multiplying $\Delta\mathscr{T} = -\delta\mathscr{T} + \mathscr{S}W$ by $(\eta\mathscr{S}W)$ and integrating by parts yields

$$\iint\limits_B \nabla\mathscr{T}\cdot\nabla(\eta\mathscr{S}W)\,dudv = \iint\limits_B \eta\,\delta\mathscr{T}\cdot\mathscr{S}W\,dudv - \iint\limits_B \eta\,|\mathscr{S}W|^2\,dudv.$$

Taking $|\delta\mathscr{T}| \le \sqrt{n-2}|\mathscr{T}_u||\mathscr{T}_v|$ (see section 8.4) into account, we can now estimate as follows

$$\begin{aligned}
\iint\limits_{B_\rho} |\mathscr{S}W|^2\,dudv &\le \iint\limits_B \eta\,|\mathscr{S}W|^2\,dudv \\
&\le \iint\limits_B \eta\,\big|\delta\mathscr{T}\cdot\mathscr{S}W\big|\,dudv + \iint\limits_B \big|\nabla\mathscr{T}\cdot\nabla(\eta\mathscr{S}W)\big|\,dudv \\
&\le \|\mathscr{S}W\|_{L^\infty(B)} \iint\limits_B \eta\,|\delta\mathscr{T}|\,dudv + \iint\limits_B |\nabla\eta|\,|\mathscr{S}W|\,|\nabla\mathscr{T}|\,dudv \\
&\quad + \iint\limits_B \eta\,|\nabla(\mathscr{S}W)|\,|\nabla\mathscr{T}|\,dudv \\
&\le \frac{\sqrt{n-2}}{2}\,\|\mathscr{S}W\|_{L^\infty(B)} \iint\limits_B |\nabla\mathscr{T}|^2\,dudv + \frac{\varepsilon}{2}\iint\limits_B |\mathscr{S}W|^2\,dudv \\
&\quad + \frac{1}{2\varepsilon(1-\rho)^2}\iint\limits_B |\nabla\mathscr{T}|^2\,dudv \\
&\quad + \frac{\delta}{2}\iint\limits_B |\nabla(\mathscr{S}W)|^2\,dudv + \frac{1}{2\delta}\iint\limits_B |\nabla\mathscr{T}|^2\,dudv
\end{aligned}$$

with arbitrary real numbers $\varepsilon, \delta > 0$.

Summarizing we arrive at

$$\|\mathscr{S}W\|^2_{L^2(B_\rho)} \le \left(\frac{\sqrt{n-2}}{2}\,\|\mathscr{S}W\|_{L^\infty(B)} + \frac{1}{2\varepsilon(1-\rho)^2} + \frac{1}{2\delta}\right)\|\nabla\mathscr{T}\|^2_{L^2(B)}$$
$$+\frac{\varepsilon}{2}\,\|\mathscr{S}W\|^2_{L^2(B)} + \frac{\delta}{2}\,\|\nabla(\mathscr{S}W)\|^2_{L^2(B)}\,.$$

2. Now we choose ε : Inserting

$$\varepsilon = \|\mathscr{S}W\|^{-2}_{L^2(B)}\|\mathscr{S}W\|^2_{L^2(B_\rho)} > 0$$

and rearranging for $\|\mathscr{S}\|^2_{L^2(B_\rho)}$ gives

$$\|\mathscr{S}W\|^2_{L^2(B_\rho)} \le \left(\sqrt{n-2}\,\|\mathscr{S}W\|_{L^\infty(B)} + \frac{\|\mathscr{S}W\|^2_{L^2(B)}}{(1-\rho)^2\|\mathscr{S}W\|^2_{L^2(B_\rho)}}\right)\|\nabla\mathscr{T}\|^2_{L^2(B)}$$
$$+\frac{1}{\delta}\,\|\nabla\mathscr{T}|^2_{L^2(B)} + \delta\|\nabla(\mathscr{S}W)\|^2_{L^2(B)}\,.$$

And since $\|\nabla(\mathscr{S}W)\|_{L^2(B)} > 0$ we can insert

$$\delta = \frac{1}{2}\,\|\nabla(\mathscr{S}W)\|^{-2}_{L^2(B)}\|\mathscr{S}W\|^2_{L^2(B_\rho)}$$

which implies

$$\|\mathscr{S}W\|^2_{L^2(B_\rho)} \le 2\left(\sqrt{n-2}\,\|\mathscr{S}W\|_{L^\infty(B)} + \frac{\|\mathscr{S}W\|^2_{L^2(B)}}{(1-\rho)^2\|\mathscr{S}W\|^2_{L^2(B_\rho)}}\right)\|\nabla\mathscr{T}\|^2_{L^2(B)}$$
$$+2\cdot\frac{2\|\nabla(\mathscr{S}W)\|^2_{L^2(B)}}{\|\mathscr{S}W\|^2_{L^2(B_\rho)}}\,\|\nabla\mathscr{T}\|^2_{L^2(B)}\,.$$

Having $\mathscr{T}_X[N] = \|\nabla\mathscr{T}\|^2_{L^2(B)}$ in mind we arrive at the stated estimate. □

8.8 Existence and regularity of weak normal Coulomb frames

Regularity results for the homogeneous Poisson problem

To introduce the function spaces coming next into play we consider the Dirichlet boundary value problem

$$\Delta\phi(u,v) = r(u,v) \quad \text{in } B, \quad \phi(u,v) = 0 \quad \text{on } \partial B. \tag{DP}$$

We want to recall very briefly some important techniques to establish existence and regularity of solutions ϕ to (DP).

Schauder estimates

Let $r \in C^{\alpha}(B,\mathbb{R})$ hold true for the right hand side r. Then there exist a classical solution of (DP) satisfying

$$\|\phi\|_{C^{2+\alpha}(B)} \le C(\alpha)\|r\|_{C^{\alpha}(B)},$$

see e.g Gilbarg and Trudinger [71].

L^p-estimates

Now let $r \in L^2(\mathring{B},\mathbb{R})$. Then any solution $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ belongs to the class $H^{2,2}(\mathring{B},\mathbb{R})$, and it holds

$$\|\phi\|_{H^{2,2}(\mathring{B})} \le C\big(\|\phi\|_{H^{1,2}(\mathring{B})} + \|r\|_{L^2(\mathring{B})}\big).$$

In particular, if $r \in H^{m-2,2}(\mathring{B},\mathbb{R})$ holds true for the right hand side we have

$$\|\phi\|_{H^{m,2}(\mathring{B})} \le C\big(\|\phi\|_{H^{1,2}(\mathring{B})} + \|r\|_{H^{m-2,2}(\mathring{B})}\big).$$

For a detailed analysis we refer the reader to Dobrowolski [46], chapter 7. Note that we must require $r \in L^2(\mathring{B},\mathbb{R})$ to infer higher regularity $\phi \in C^0(\mathring{B},\mathbb{R})$ because $H^{2,2}(\mathring{B},\mathbb{R})$ is continuously emedded in $C^0(\mathring{B},\mathbb{R})$ by Sobolev's embedding theorem.

If on the other hand $r \in L^1(\mathring{B},\mathbb{R})$ then any weak solution $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ of (DP) satisfies

$$\|\phi\|_{L^q(\mathring{B})} \le C\|r\|_{L^1(\mathring{B})} \quad \text{for all } 1 \le q < \infty,$$

$$\|\phi\|_{H^{1,p}(\mathring{B})} \le C\|r\|_{L^1(\mathring{B})} \quad \text{for all } 1 \le p < 2.$$

A function $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ is not necessarily continuous.

Wente's L^∞-estimate

This situation changes dramatically if the right hand side r possesses certain algebraic structures. In particular, assume that

$$r = \frac{\partial a}{\partial u}\frac{\partial b}{\partial v} - \frac{\partial a}{\partial v}\frac{\partial b}{\partial u}$$

with functions $a, b \in H^{1,2}(\mathring{B},\mathbb{R})$.

Then again $r \in L^1(\mathring{B},\mathbb{R})$, but any solution $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ is of class $C^0(B,\mathbb{R})$ and satisfies Wente's L^∞-estimate

$$\|\phi\|_{L^\infty(\mathring{B})} + \|\nabla\phi\|_{L^2(\mathring{B})} \le \frac{1}{4\pi}\,\|\nabla a\|_{L^2(\mathring{B})}\|\nabla b\|_{L^2(\mathring{B})}\,,$$

see Wente [166]. We already used this inequality in section 8.6 for establishing an upper bound for the functional of total torsion of normal Coulomb frames.

Hardy spaces

Wente's discovery is the starting point of the modern harmonic analysis. Its general framework is the concept of *Hardy spaces* $\mathscr{H}^1(\mathbb{R}^m,\mathbb{R})$.

From Helein [82] we quote two common definitions of Hardy spaces which finally lead to equivalent formulations of the Hardy space theory.

Definition 8.2. (Tempered-distribution definition)
Let $\Psi \in C_0^\infty(\mathbb{R}^m,\mathbb{R})$ such that

$$\int\limits_{\mathbb{R}^m} \Psi(x)\,dx = 1.$$

For each $\varepsilon > 0$ we set

$$\Psi_\varepsilon(x) = \frac{1}{\varepsilon^m}\,\Psi(\varepsilon^{-1}x),$$

and for $\phi \in L^1(\mathbb{R}^m,\mathbb{R})$ define

$$\phi^*(x) = \sup_{\varepsilon>0}|(\Psi_\varepsilon \star \phi)(x)|.$$

Then ϕ belongs to $\mathscr{H}^1(\mathbb{R}^m,\mathbb{R})$ if and only if $\phi^* \in L^1(\mathbb{R}^m,\mathbb{R})$ with norm

$$\|\phi\|_{\mathscr{H}^1(\mathbb{R}^m,\mathbb{R})} \le \|\phi\|_{L^1(\mathbb{R}^m)} + \|\phi^*\|_{L^1(\mathbb{R}^m)}\,.$$

Definition 8.3. (Riesz-Fourier-transform definition)
For any function $\phi \in L^1(\mathbb{R}^m,\mathbb{R})$ we denote by $R_\alpha\phi$ the function defined by

$$\mathscr{F}(R_\alpha\phi) = \frac{\xi_\alpha}{|\xi|}\,\mathscr{F}(\phi)(\xi)$$

with the ϕ-Fourier transform

$$\mathscr{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{m}{2}}}\int\limits_{\mathbb{R}^m} e^{-ix\cdot\xi}\,\phi(x)\,dx.$$

Then ϕ *belongs to* $\mathscr{H}^1(\mathbb{R}^m,\mathbb{R})$ *if and only if*

$$R_\alpha\phi \in L^1(\mathbb{R}^m,\mathbb{R}) \quad \text{for all } \alpha = 1,\ldots,m$$

with norm

$$\|\phi\|_{\mathscr{H}^1(\mathbb{R}^m)} = \|\phi\|_{L^1(\mathbb{R}^m)} + \sum_{\alpha=1}^{m} \|R_\alpha\phi\|_{L^1(\mathbb{R}^m)}\,.$$

For a comprehensive presentation of harmonic analysis we would also like to refer the reader to Stein's monograph [151].

Consider again our Dirichlet problem (DP). Let $a,b \in H^{1,2}(\mathring{B},\mathbb{R})$, and consider its extensions $a \mapsto \widehat{a}$ and $b \mapsto \widehat{b}$ in $H^{1,2}(\mathbb{R}^2,\mathbb{R})$ to the whole space $\mathbb{R}^2$ such that these mappings are continuous from $H^{1,2}(\mathring{B},\mathbb{R})$ to $H^{1,2}(\mathbb{R}^2,\mathbb{R})$. *Then due to Helein [82] it holds* $r \in \mathscr{H}^1(\mathbb{R}^2,\mathbb{R})$.

Lorentz interpolation spaces

This latter fact becomes especially important in the following. Let us start with

Definition 8.4. Let $p \in (1,+\infty)$ and $q \in [1,+\infty]$. The *Lorentz space* $L^{(p,q)}(\mathring{B},\mathbb{R})$ is the set of measurable functions $\phi : \mathring{B} \to \mathbb{R}$ such that

$$\|f\|_{L^{(p,q)}(\mathring{B})} := \left(\int_0^\infty \left\{ t^{\frac{1}{p}}\phi^*(t) \right\}^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \quad \text{if } q < +\infty$$

or

$$\|f\|_{L^{(p,q)}(\mathring{B})} := \sup_{t>0} t^{\frac{1}{p}}\phi^*(t) < \infty \quad \text{if } q = +\infty.$$

Here ϕ^* denotes the unique non-increasing rearrangement of $|\phi|$ on $[0, \text{meas}\,(\mathring{B})]$.

Lorentz spaces are Banach spaces with a suitable norm. They may be considered as a deformation of L^p.

Notice that

$$L^{(p,p)}(\mathring{B},\mathbb{R}) = L^p(\mathring{B},\mathbb{R})\,,$$

$$L^{(p,1)}(\mathring{B},\mathbb{R}) \subset L^{(p,q')}(\mathring{B},\mathbb{R}) \subset L^{(p,q'')}(\mathring{B},\mathbb{R}) \subset L^{(p,\infty)}(\mathring{B},\mathbb{R})$$

for $1 < q' < q''$. Then

(i) if $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ solves (DP) with $r \in \mathscr{H}^1(\mathring{B},\mathbb{R})$ then $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \in L^{(2,1)}(\mathring{B},\mathbb{R})$;
(i) if $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ with $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \in L^{(2,1)}(\mathring{B},\mathbb{R})$ then $\phi \in C^0(\mathring{B},\mathbb{R})$.

The general regularity result

Summarizing the foregoing facts we can state the following regularity result from Helein [82], chapter 3.

Proposition 8.9. *Let* $a,b \in H^{1,2}(\mathring{B},\mathbb{R})$, *and assume* $\phi \in H^{1,2}(\mathring{B},\mathbb{R})$ *solves (DP). Then* $\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v} \in L^{(2,1)}(\mathring{B},\mathbb{R})$, *and in particular* $\phi \in C^0(B,\mathbb{R})$.

Existence of weak normal Coulomb frames

In case $n=2$ we constructed critical points N of the functional $\mathscr{T}_X[N]$ of total torsion by solving the Euler-Lagrange equation explicitely. In the general situation here we construct critical points by means of direct methods of the calculus of variations.

We start with the following

Definition 8.5. Let $m \in \mathbb{N}$, $m \geq 2$. For two matrices $\mathbf{A},\mathbf{B} \in \mathbb{R}^{m\times m}$ we define their *inner product*

$$\langle \mathbf{A},\mathbf{B}\rangle = \operatorname{trace}(\mathbf{A}\circ\mathbf{B}^T) = \sum_{\sigma,\vartheta=1}^{m} a_{\sigma\vartheta}b_{\sigma\vartheta}$$

and *the associated norm*

$$|\mathbf{A}| := \sqrt{\langle \mathbf{A},\mathbf{A}\rangle} = \left(\sum_{\sigma,\vartheta=1}^{m} a_{\sigma\vartheta}^2\right)^{\frac{1}{2}}.$$

Helein proved in [82], Lemma 4.1.3, *existence of weak Coulomb frames in the tangent bundle of surfaces.* We want to carry out his arguments and adapt his methods to our present situation.

We consider conformally parametrized immersions

$$X = X(u,v) \quad \text{of regularity class } C^{k-1,\alpha}(B,\mathbb{R}^{n+2})$$

with some Hölder exponent $\alpha \in (0,1)$.

Proposition 8.10. *Let the conformally parametrized immersion* $X \in C^{k,\alpha}(B,\mathbb{R}^{n+2})$ *be given with some* $\alpha \in (0,1)$. *Then there exists a weak normal Coulomb frame* $N \in H^{1,2}(\mathring{B}) \cap L^\infty(\mathring{B})$ *minimizing the functional* $\mathscr{T}_X[N]$ *of total torsion in the set of all weak normal frames of class* $H^{1,2}(\mathring{B}) \cap L^\infty(\mathring{B})$.

Proof. We fix[5] some $C^{k-1,\alpha}$-regular orthonormal normal frame $\widetilde{N}$ and interpret $\mathscr{T}_X[N]$ as a functional $\mathscr{F}_X[\mathbf{R}]$ of $SO(n)$-regular rotations

$$\mathbf{R} = (r_{\sigma\vartheta})_{\sigma,\vartheta=1,\ldots,n} \in H^{1,2}(\mathring{B},SO(n))$$

[5] Notice that now we start from $\widetilde{N}$ and transform into N.

by setting

$$\mathscr{F}_X[\mathbf{R}] := \frac{1}{2}\sum_{\sigma,\vartheta=1}^{n}\sum_{i=1}^{2}\iint_B (T_{\sigma,i}^{\vartheta})^2\,dudv = \frac{1}{2}\iint_B \big(|\mathbf{T}_1|^2+|\mathbf{T}_2|^2\big)\,dudv$$

where again

$$N_\sigma := \sum_{\vartheta=1}^{n} r_{\sigma\vartheta}\widetilde{N}_\vartheta$$

as well as

$$\mathbf{T}_i = (T_{\sigma,i}^{\vartheta})_{\sigma,\vartheta=1,\ldots,n} \in \mathbb{R}^{n\times n}\,.$$

Choose a minimizing sequence ${}^\ell\mathbf{R} = ({}^\ell r_{\sigma\vartheta})_{\sigma,\vartheta=1,\ldots,n} \in H^{1,2}(\mathring{B},SO(n))$ and define

$${}^\ell N_\sigma := \sum_{\vartheta=1}^{n} {}^\ell r_{\sigma\vartheta}\widetilde{N}_\vartheta\,.$$

As in section 6.3 we find[6]

$${}^\ell\mathbf{T}_i = {}^\ell\mathbf{R}_{u^i}\circ{}^\ell\mathbf{R}^T + {}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i\circ{}^\ell\mathbf{R}^T$$

which implies

$$\begin{aligned}{}^\ell\mathbf{T}_i\circ{}^\ell\mathbf{T}_i^T &= ({}^\ell\mathbf{R}_{u^i}\circ{}^\ell\mathbf{R}^T + {}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i\circ{}^\ell\mathbf{R}^T)\circ({}^\ell\mathbf{R}\circ{}^\ell\mathbf{R}_{u^i}^T + {}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i^T\circ{}^\ell\mathbf{R}^T)\\ &= {}^\ell\mathbf{R}_{u^i}\circ{}^\ell\mathbf{R}_{u^i}^T + {}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i\circ{}^\ell\mathbf{R}_{u^i}^T + {}^\ell\mathbf{R}_{u^i}\circ\widetilde{\mathbf{T}}_i^T\circ{}^\ell\mathbf{R}^T + {}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i\circ\widetilde{\mathbf{T}}_i^T\circ{}^\ell\mathbf{R}^T\,.\end{aligned}$$

In particular, we conclude

$$\operatorname{trace}({}^\ell\mathbf{T}_i\circ{}^\ell\mathbf{T}_i^T) = \operatorname{trace}({}^\ell\mathbf{R}_{u^i}\circ{}^\ell\mathbf{R}_{u^i}^T) + 2\operatorname{trace}({}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i\circ{}^\ell\mathbf{R}_{u^i}^T) + \operatorname{trace}(\widetilde{\mathbf{T}}_i\circ\widetilde{\mathbf{T}}_i^T)$$

or, using our notion of a matrix norm,

$$|{}^\ell\mathbf{T}_i|^2 = |{}^\ell\mathbf{R}_{u^i}|^2 + 2\langle{}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i,{}^\ell\mathbf{R}_{u^i}\rangle + |\widetilde{\mathbf{T}}_i|^2. \tag{8.3}$$

Furthermore, taking $|{}^\ell\mathbf{R}\circ\widetilde{\mathbf{T}}_i| = |\widetilde{\mathbf{T}}_i|$ into account, we arrive at the estimate

$$|{}^\ell\mathbf{T}_i|^2 \ge \big(|\widetilde{\mathbf{T}}_i| - |{}^\ell\mathbf{R}_{u^i}|\big)^2 \quad \text{a.e. on } B, \quad \text{for all } \ell\in\mathbb{N}.$$

Now because the $\widetilde{\mathbf{T}}_i$ are bounded in $L^2(\mathring{B},\mathbb{R})$, and since ${}^\ell\mathbf{R}$ is minimizing for $\mathscr{F}_X[\mathbf{R}]$, the sequences ${}^\ell\mathbf{T}_i$ are also bounded in $L^2(\mathring{B},\mathbb{R})$. Thus ${}^\ell\mathbf{R}_{u^i}$ are bounded sequences in $L^2(\mathring{B},SO(n))$ in accordance with the last inequality.

---

[6] Note that the proof of this identity remains true for $R\in H^{2,1}(\mathring{B},SO(n))\cap H^{1,2}(\mathring{B},SO(n))$.

By Hilbert's selection theorem and Rellich's embedding theorem we find a subsequence, again denoted by ${}^{\ell}\mathbf{R}$, which converges as follows:

$$ {}^{\ell}\mathbf{R}_{u^i} \rightharpoonup \mathbf{R}_{u^i} \quad \text{weakly in } L^2(\mathring{B}, SO(n)), \quad {}^{\ell}\mathbf{R} \to \mathbf{R} \quad \text{strongly in } L^2(\mathring{B}, SO(n)) $$

with some $\mathbf{R} \in H^{1,2}(\mathring{B}, SO(n))$. In particular, going if necessary to a subsequence, we have ${}^{\ell}\mathbf{R} \to \mathbf{R}$ a.e. on $\mathring{B}$ and

$$ \lim_{\ell\to\infty} \iint_B |{}^{\ell}\mathbf{R} \circ \widetilde{\mathbf{T}}_i - \mathbf{R} \circ \widetilde{\mathbf{T}}_i|^2 \, dudv = 0 $$

according to the dominated convergence theorem. Hence we can compute in the limit

$$ \begin{aligned} &\lim_{\ell\to\infty} \iint_B \langle {}^{\ell}\mathbf{R} \circ \widetilde{\mathbf{T}}_i, {}^{\ell}\mathbf{R}_{u^i} \rangle \, dudv \\ &\quad = \lim_{\ell\to\infty} \left(\iint_B \langle {}^{\ell}\mathbf{R} \circ \widetilde{\mathbf{T}}_i - \mathbf{R} \circ \widetilde{\mathbf{T}}_i, {}^{\ell}\mathbf{R}_{u^i} \rangle \, dudv + \iint_B \langle \mathbf{R} \circ \widetilde{\mathbf{T}}_i, {}^{\ell}\mathbf{R}_{u^i} \rangle \, dudv \right) \\ &\quad = \iint_B \langle \mathbf{R} \circ \widetilde{\mathbf{T}}_i, \mathbf{R}_{u^i} \rangle \, dudv. \end{aligned} $$

In addition we obtain

$$ \lim_{\ell\to\infty} \iint_B |{}^{\ell}\mathbf{R}_{u^i}|^2 \, dudv \geq \iint_B |\mathbf{R}_{u^i}|^2 \, dudv $$

due to the semicontinuity of the L^2-norm w.r.t. weak convergence. Putting the last two relations into the identity

$$ |{}^{\ell}\mathbf{T}_i|^2 = |{}^{\ell}\mathbf{R}_{u^i}|^2 + 2\langle {}^{\ell}\mathbf{R} \circ \widetilde{\mathbf{T}}_i, {}^{\ell}\mathbf{R}_{u^i} \rangle + |\widetilde{\mathbf{T}}_i|^2 $$

we finally infer

$$ \begin{aligned} \lim_{\ell\to\infty} \mathscr{F}_X[{}^{\ell}\mathbf{R}] &= \frac{1}{2} \lim_{\ell\to\infty} \iint_B \left(|{}^{\ell}\mathbf{T}_1|^2 + |{}^{\ell}\mathbf{T}_2|^2 \right) dudv \\ &\geq \frac{1}{2} \iint_B \left(|\mathbf{R}_u|^2 + |\mathbf{R}_v|^2 \right) dudv + \iint_B \left(\langle \mathbf{R} \circ \widetilde{\mathbf{T}}_1, \mathbf{R}_u \rangle + \langle \mathbf{R} \circ \widetilde{\mathbf{T}}_2, \mathbf{R}_v \rangle \right) dudv \\ &\quad + \frac{1}{2} \iint_B \left(|\widetilde{\mathbf{T}}_1|^2 + |\widetilde{\mathbf{T}}_2|^2 \right) dudv \\ &= \frac{1}{2} \iint_B \left(|\mathbf{T}_1|^2 + |\mathbf{T}_2|^2 \right) dudv = \mathscr{F}_X[\mathbf{R}] \end{aligned} $$

where $\mathbf{T}_i = (T_{\sigma,i}^{\vartheta})_{\sigma,\vartheta=1,\dots,n}$ denote the torsion coefficients of the frame N with entries $N_\sigma := \sum_\vartheta r_{\sigma\vartheta}\widetilde{N}_\vartheta$. Consequently,

$$N \in H^{1,2}(\mathring{B}) \cap L^\infty(\mathring{B})$$

minimizes $\mathscr{T}_X[N]$ and, in particular, it is a weak normal Coulomb frame. □

$H^{2,1}_{loc}$-regularity of weak normal Coulomb frames

To prove higher regularity of normal Coulomb frames we make essential use of techniques from harmonic analysis. Consult eventually paragraph 8.8. We always use conformal parameters $(u,v) \in B$.

Proposition 8.11. *Any weak normal Coulomb frame $N \in H^{1,2}(\mathring{B}) \cap L^\infty(\mathring{B})$ for the conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ belongs to the class $H^{2,1}_{loc}(\mathring{B})$.*

Proof. 1. Following section 8.2 the torsion coefficients $T_{\sigma,i}^{\vartheta}$, $\sigma,\vartheta = 1,\dots,n$, of the normal Coulomb frame N are weak solutions of the Euler-Lagrange equations

$$\operatorname{div}(T_{\sigma,1}^{\vartheta}, T_{\sigma,2}^{\vartheta}) = 0 \quad \text{in } \mathring{B}.$$

Hence by a weak version of Poincare's lemma (see e.g. Bourgain, Brezis and Mironescu [17] Lemma 3), there exist integral functions $\partial_u \tau^{(\sigma\vartheta)} \in H^{1,2}(B)$ satisfying

$$\partial_u \tau^{(\sigma\vartheta)} = -T_{\sigma,2}^{\vartheta}\,, \quad \partial_v \tau^{(\sigma\vartheta)} = T_{\sigma,1}^{\vartheta} \quad \text{in } \mathring{B}$$

Thus the weak form of the Euler-Lagrange equations can be written as

$$0 = \iint\limits_B \left\{\varphi_u \partial_v \tau^{(\sigma\vartheta)} - \varphi_v \partial_u \tau^{(\sigma\vartheta)}\right\} dudv = \int\limits_{\partial B} \tau^{(\sigma\vartheta)} \frac{\partial \varphi}{\partial t}\, ds$$

for all $\varphi \in C^\infty(B,\mathbb{R})$ where $\frac{\partial\varphi}{\partial t}$ denotes the tangential derivative of φ along ∂B. Note that $\tau^{(\sigma\vartheta)}|_{\partial B}$ means the L^2-trace of $\tau^{(\sigma\vartheta)}$ on the boundary curve ∂B (see e.g. Alt [3], chapter 6, appendix A6.6)[7]. Consequently, the lemma of DuBois-Reymond[8] yields $\tau^{(\sigma\vartheta)} \equiv \text{const}$ on ∂B, and by translation we arrive at the boundary conditions

$$\tau^{(\sigma\vartheta)} = 0 \quad \text{on } \partial B\,.$$

---

[7] Let $1 \le p \le \infty$. Then there is a uniquely determined map $S\colon H^{1,p}(\mathring{B},\mathbb{R}) \to L^p(\mathring{B},\mathbb{R})$ such that $\|S(\phi)\|_{L^p(\partial B)} \le C\|\phi\|_{H^{1,2}(\mathring{B})}$. Additionally it holds $S(\phi) = \phi\big|_{\partial B}$ if $\phi \in H^{1,2}(\mathring{B},\mathbb{R}) \cap C^0(B,\mathbb{R})$). The map S is called the trace mapping.

[8] Its one-dimensional version is the following: Let $f \in L^1((a,b),\mathbb{R})$ and assume that $\int_a^b f(x)\varphi(x)\,dx = 0$ for all $\varphi \in C^\infty([a,b],\mathbb{R})$. Then it holds $f \equiv \text{const}$ almost everywhere.

2. Thus the integral functions $\tau^{(\sigma\vartheta)}$ are weak solutions of the second-order system

$$\Delta\tau^{(\sigma\vartheta)} = -T^{\vartheta}_{\sigma,2,u} + T^{\vartheta}_{\sigma,1,v} = -N_{\sigma,v}\cdot N_{\vartheta,u} + N_{\sigma,u}\cdot N_{\vartheta,v} \quad \text{in } \mathring{B}$$

where the second identity follows by direct differentiation. By a result Coifman, Lions, Meyer and Semmes [34],[9] the right-hand side of div-curl type belongs to the Hardy space $\mathscr{H}^1_{loc}(\mathring{B},\mathbb{R})$ and, hence, the $\tau^{(\sigma\vartheta)}$ belong to $H^{2,1}_{loc}(\mathring{B},\mathbb{R})$ by a result of Fefferman and Stein[name]Stein, E.M. [55].[10] Consequently we find $T^{\vartheta}_{\sigma,i}\in H^{1,1}_{loc}(\mathring{B},\mathbb{R})\cap L^2(\mathring{B},\mathbb{R})$. Next, we employ the Weingarten equations

$$N_{\sigma,u^i} = -\sum_{j,k=1}^{2} L_{\sigma,ij}g^{jk}X_{u^k} + \sum_{\vartheta=1}^{n} T^{\vartheta}_{\sigma,i}N_{\vartheta}$$

in a weak form. For the coefficients $L_{\sigma,ij}$ of the second fundamental form we have $L_{\sigma,ij} = N_\sigma\cdot X_{u^iu^j}$ leading to $L_{\sigma,ij}\in H^{1,2}(\mathring{B},\mathbb{R})$ and taking account of $N\in H^{1,2}(\mathring{B})\cap L^\infty(\mathring{B})$ and $X_{u^iu^j}\in L^\infty(\mathring{B},\mathbb{R}^{n+2})$. Hence we arrive at $N_{\sigma,u^i}\in H^{1,1}_{loc}(\mathring{B},\mathbb{R}^{n+2})$ and $N\in H^{2,1}_{loc}(\mathring{B})$ for our weak normal Coulomb frame. Note that $T^{\vartheta}_{\sigma,i}\in H^{1,1}_{loc}(\mathring{B},\mathbb{R})\cap L^2(\mathring{B},\mathbb{R})$ and $N_\vartheta\in H^{1,2}(\mathring{B},\mathbb{R}^{n+2})\cap L^\infty(\mathring{B},\mathbb{R}^{n+2})$ imply $T^{\vartheta}_{\sigma,i}N_\vartheta\in H^{1,1}_{loc}(\mathring{B},\mathbb{R}^{n+2})$ by a careful adaption of the classical product rule in Sobolev spaces which is explained in the lemma following immediately. We have proved the statement. □

Lemma 8.2. *Under the assumptions of the foregoing proposition it holds*

$$\sum_{\vartheta=1}^{n} T^{\vartheta}_{\sigma,i}N_\vartheta \in H^{1,1}(\mathring{B},\mathbb{R}^{n+2}).$$

Proof. Note that $T^{\vartheta}_{\sigma,i}N_\vartheta\in L^2(\mathring{B},\mathbb{R}^{n+2})$ since $T^{\vartheta}_{\sigma,i}\in L^2(\mathring{B},\mathbb{R})$ and $N_\vartheta\in L^\infty(\mathring{B},\mathbb{R}^{n+2})$. We show that $T^{\vartheta}_{\sigma,i}N_\vartheta$ has a weak derivative, i.e. we prove that

$$T^{\vartheta}_{\sigma,i}N_{\vartheta,u^j} + N_\vartheta\partial_{u^j}T^{\vartheta}_{\sigma,i}\in L^1(\mathring{B},\mathbb{R}^{n+2})$$

is the weak derivative of $T^{\vartheta}_{\sigma,i}N_\vartheta$. In other words

$$\iint_B (T^{\vartheta}_{\sigma,i}N_{\vartheta,u^j} + N_\vartheta\partial_{u^j}T^{\vartheta}_{\sigma,i})\varphi\,dudv = -\iint_B (T^{\vartheta}_{\sigma,i}N_\vartheta)\varphi_{u^j}\,dudv$$

for all $\varphi\in C^\infty_0(B,\mathbb{R})$. For such a test function φ define

$$\psi := T^{\vartheta}_{\sigma,i}\varphi \in W^{1,1}_0(\mathring{B},\mathbb{R})\cap L^2(\mathring{B},\mathbb{R}).$$

---

[9] Let $\phi: H^{1,2}(\mathbb{R}^2,\mathbb{R})$. Then $f := \det(\nabla\phi)\in\mathscr{H}^1(\mathbb{R}^2,\mathbb{R})$, and it holds $\|f\|_{\mathscr{H}^1(\mathbb{R}^2)}\le C\|\phi\|_{H^{1,2}(\mathbb{R}^2)}$.

[10] Let $\phi\in L^1(\mathbb{R}^2,\mathbb{R})$ be a solution of $-\Delta\phi = f\in\mathscr{H}^1(\mathbb{R}^2,\mathbb{R})$. Then all second derivatives of ϕ belong to $L^1(\mathbb{R}^2,\mathbb{R})$, and it holds $\|\phi_{x^\alpha x^\beta}\|_{L^1(\mathbb{R}^m)}\le C\|f\|_{L^1(\mathbb{R}^m)}$ for all $\alpha,\beta = 1,2$.

We claim

$$\iint_B N_{u^j}\,\psi\,dudv = -\iint_B N\,\psi_{u^j}\,dudv.$$

For the proof of this relation we approximate ψ with smooth functions $\psi^\varepsilon \in C_0^\infty(B,\mathbb{R})$ in the sense of Friedrichs. Then $\psi^\varepsilon \to \psi$ in $H^{1,1}(\mathring{B},\mathbb{R}) \cap L^2(\mathring{B},\mathbb{R})$, and $\psi = 0$ outside some compact set $K \subset\subset \mathring{B}$. We estimate as follows

$$\begin{aligned}
&\left|\iint_B (N_{\vartheta,u^j}\,\psi + N_\vartheta\,\psi_{u^j})\,dudv\right| \\
&\quad = \left|\iint_B (N_{\vartheta,u^j}\,\psi^\varepsilon + N_\vartheta\,\psi^\varepsilon_{u^j})\,dudv\right| + \left|\iint_B N_{\vartheta,u^j}\,(\psi - \psi^\varepsilon)\,dudv\right| \\
&\qquad + \left|\iint_B N_\vartheta\,(\psi^\varepsilon_{u^j} - \psi_{u^j})\,dudv\right| \\
&\quad \le \|N_{\vartheta,u^j}\|_{L^2(\mathring{B})}\|\psi - \psi^\varepsilon\|_{L^2(\mathring{B})} + \|N_\vartheta\|_{L^\infty(\mathring{B})}\|\psi^\varepsilon_{u^j} - \psi_{u^j}\|_{L^1(\mathring{B})}
\end{aligned}$$

taking

$$\iint_B N_{u^j}\,\psi^\varepsilon\,dudv = -\iint_B N\,\psi^\varepsilon_{u^j}\,dudv$$

into account. Since $\|\psi - \psi^\varepsilon\|_{L^2(\mathring{B})} \to 0$ and $\|\psi^\varepsilon_{u^j} - \psi_{u^j}\|_{L^1(\mathring{B})} \to 0$ for $\varepsilon \to 0$ we arrive at the stated identity. Now we use the product rule and calculate

$$\begin{aligned}
\iint_B (T^\vartheta_{\sigma,i}N_{\vartheta,u^j} + N_\vartheta\,\partial_{u^j}T^\vartheta_{\sigma,i})\varphi\,dudv &= \iint_B N_{\vartheta,u^j}\,\psi\,dudv + \iint_B N_\vartheta\,\partial_{u^j}T^\vartheta_{\sigma,i}\varphi\,dudv \\
&= -\iint_B (T^\vartheta_{\sigma,i}\varphi)_{u^j}N_\vartheta\,dudv + \iint_B N_\vartheta\,\partial_{u^j}T^\vartheta_{\sigma,i}\varphi\,dudv \\
&= -\iint_B \partial_{u^j}T^\vartheta_{\sigma,i}N_\vartheta\,\varphi\,dudv - \iint_B T^\vartheta_{\sigma,i}\varphi_{u^j}N\,dudv + \iint_B N_\vartheta\,\partial_{u^j}T^\vartheta_{\sigma,i}\varphi\,dudv \\
&= -\iint_B (T^\vartheta_{\sigma,i}N_\vartheta)\varphi_{u^j}\,dudv.
\end{aligned}$$

This proves the statement. □

8.9 Classical regularity of normal Coulomb frames

Our main result of this chapter is a proof of classical regularity of normal Coulomb frames. Up to now we only know regularity in the sense of Sobolev spaces. An essential tool on our road to regularity are again the Weingarten equations.

Theorem 8.3. *For any conformally parametrized immersion $X \in C^{k,\alpha}(B,\mathbb{R}^{n+2})$ with $k \geq 3$ and $\alpha \in (0,1)$ there exists a normal Coulomb frame $N \in C^{k-1,\alpha}(B)$ minimizing the functional $\mathscr{T}_X[N]$ of total torsion.*

Proof. 1. We fix some normal frame $\widetilde{N} \in C^{k-1,\alpha}(B)$, and as above we construct a weak normal Coulomb frame N from the regularity class $H^{1,2}(\mathring{B}) \cap L^\infty(\mathring{B})$. Furthermore we then know $N \in H^{2,1}_{loc}(\mathring{B})$. Defining the orthogonal mapping $\mathbf{R} = (r_{\sigma\vartheta})_{\sigma,\vartheta=1,\ldots,n}$ by $r_{\sigma\vartheta} := N_\sigma \cdot \widetilde{N}_\vartheta$ we thus find

$$N_\sigma = \sum_{\vartheta=1}^{n} r_{\sigma\vartheta}\widetilde{N}_\vartheta \quad \text{and} \quad \mathbf{R} \in H^{2,1}_{loc}(\mathring{B},SO(n)) \cap H^{1,2}(\mathring{B},SO(n))\,.$$

In particular, we can assign a normal curvature matrix $\mathbf{S}_{12} = (S^{\vartheta}_{\sigma,12})_{\sigma,\vartheta=1,\ldots,n} \in L^1_{loc}(\mathring{B},\mathbb{R}^{n\times n})$ to N by the formula

$$S^{\vartheta}_{\sigma,12} = \partial_v T^2_{\sigma,1} - \partial_u T^{\vartheta}_{\sigma,2} + \sum_{\omega=1}^{n} (T^{\omega}_{\sigma,1}T^{\vartheta}_{\omega,2} - T^{\omega}_{\sigma,2}T^{\vartheta}_{\omega,1}).$$

From the previous section we then infer $\mathbf{S}_{12} \in L^\infty(\mathring{B},\mathbb{R}^{n\times n})$.

2. Introduce $\tau = (\tau^{(\sigma\vartheta)})_{\sigma,\vartheta=1,\ldots,n} \in H^{1,2}(\mathring{B},\mathbb{R}^{n\times n})$ by

$$\begin{aligned}
&\partial_u \tau^{(\sigma\vartheta)} = -T^{\vartheta}_{\sigma,2}\,, \quad \partial_v \tau^{(\sigma\vartheta)} = T^{\vartheta}_{\sigma,1} \quad \text{in } \mathring{B},\\
&\tau^{(\sigma\vartheta)} = 0 \quad \text{on } \partial B.
\end{aligned}$$

The above definition of the normal curvature marix gives us the nonlinear elliptic system

$$\begin{aligned}
\Delta\tau^{(\sigma\vartheta)} &= -\sum_{\omega=1}^{n} (\partial_u \tau^{(\sigma\omega}\partial_v \tau^{(\omega\vartheta)} - \partial_v \tau^{(\sigma\omega)}\partial_u \tau^{(\omega\vartheta)}) + S^{\vartheta}_{\sigma,12} \quad \text{in } \mathring{B},\\
\tau^{(\sigma\vartheta)} &= 0 \quad \text{on } \partial B.
\end{aligned}$$

On account of $\mathbf{S}_{12} = (S^{\vartheta}_{\sigma,12})_{\sigma,\vartheta=1,\ldots,n} \in L^\infty(\mathring{B},\mathbb{R}^{n\times n})$, a part of Wente's inequality yields $\tau \in C^0(B,\mathbb{R}^{n\times n})$, see e.g. Brezis and Coron [19]; compare also Rivière [136] and the corresponding boundary regularity theorem in Müller and Schikorra [124] for more general results. By appropriate reflection of τ and $\mathbf{S}_{12}$ (the reflected quantities are again denoted by τ and $\mathbf{S}_{12}$) we obtain a weak solution

$\tau \in H^{1,2}(\mathring{B}_{1+d},\mathbb{R}^{n\times n})\cap C^0(\mathring{B}_{1+d},\mathbb{R}^{n\times n})$ of

$$\Delta\tau = f(w,\nabla\tau) \quad \text{in } \mathring{B}_{1+d} := \{w\in\mathbb{R}^2 : |w|<1+d\}$$

with some $d>0$ and a right-hand side f satisfying

$$|f(w,p)| \le a|p|^2+b \quad \text{for all } p\in\mathbb{R}^{2n^2}, \quad w\in\mathring{B}_{1+d}$$

with some reals $a,b>0$. Now applying Tomi's regularity result from [157] for weak solutions of this non-linear system possessing small variation locally in $\mathring{B}_{1+d}$, we find $\tau\in C^{1,\nu}(B,\mathbb{R}^{n\times n})$ for any $\nu\in(0,1)$ (note that Tomi's result applies for such systems with $b=0$, but his proof can easily be adapted to our inhomogeneous case $b>0$).

3. From the first-order system for $\tau^{(\sigma\vartheta)}$ we infer $\mathbf{T}_i\in C^\alpha(B,\mathbb{R}^{n\times n})$. Thus the Weingarten equations for N_{σ,u^i} yield $N\in W^{1,\infty}(\mathring{B})$ on account of $N\in L^\infty(\mathring{B})$, and we obtain $N\in C^\alpha(B)$ by Sobolev's embedding theorem. Inserting this again into the Weingarten equations we find $N\in C^{1,\alpha}(B)$. Hence we conclude $\mathbf{R}=(N_\sigma\cdot\widetilde{N}_\vartheta)_{\sigma,\vartheta=1,\ldots,n}\in C^{1,\alpha}(B,\mathbb{R}^{n\times n})$, and the transformation rule $\mathbf{S}_{12}=\mathbf{R}\circ\widetilde{\mathbf{S}}_{12}\circ\mathbf{R}^t$ implies $\mathbf{S}_{12}=(S_{\sigma,12}^\vartheta)_{\sigma,\vartheta=1,\ldots,n}\in C^\alpha(B,\mathbb{R}^{n\times n})$ (note $\widetilde{\mathbf{S}}_{12}\in C^\alpha(B,\mathbb{R}^{n\times n})$ for $k=3$; in case $k\ge 4$ we even get $\mathbf{S}_{12}\in C^{1,\alpha}(B,\mathbb{R}^{n\times n})$). Now the right-hand side of the equation for $\Delta\tau^{(\sigma\vartheta)}$ belongs to $C^\alpha(B,\mathbb{R})$, and potential theoretic estimates ensure $\tau\in C^{2,\alpha}(B,\mathbb{R}^{n\times n})$. Involving again our first-order system for the $\tau^{(\sigma\vartheta)}$ gives $\mathbf{T}_i\in C^{1,\alpha}(B,\mathbb{R}^{n\times n})$ which proves $N\in C^{2,\alpha}(B)$ using the Weingarten equations once more. Finally, for $k\ge 4$, we can bootstrap by concluding $\mathbf{R}\in C^{2,\alpha}(B,SO(n))$ and $\mathbf{S}_{12}\in C^{1,\alpha}(B)$ from the transformation rule for $\mathbf{S}_{12}$ and repeating the arguments above. This completes the proof. □

Chapter 9
Minimal surfaces

In this chapter we consider minimal surfaces as critical points of the area functional in parametric and non-parametric form. We investigate analytical and geometric properties of these immersions, and we present various proofs for establishing curvature estimates and theorems of Bernstein type.

9.1 Minimal surfaces

Minimal surfaces are critical points of the area functional

$$\mathscr{A}[X] := \iint_B W\,dudv$$

which implies their characteristic property $H \equiv 0$ as the following theorem shows.

Theorem 9.1. *The functional*

$$\delta_{\widehat{N}}\mathscr{A}[X;\varphi] := \frac{d}{d\varepsilon}\,\mathscr{A}[X+\varepsilon\varphi\widehat{N}]\Big|_{\varepsilon=0}$$

of the first variation of $\mathscr{A}[X]$ w.r.t. perturbations

$$\widetilde{X}(u,v) = X(u,v) + \varepsilon\varphi(u,v)\widehat{N}(u,v)$$

with an arbitrary unit normal vector $\widehat{N}$, a smooth function $\varphi \in C_0^\infty(B,\mathbb{R})$ and a parameter $\varepsilon \in (-\varepsilon_0, +\varepsilon_0)$ is given by

$$\delta_{\widehat{N}}\mathscr{A}[X;\varphi] = -2\iint_B H_{\widehat{N}}W\varphi\,dudv$$

with the mean curvature $H_{\widehat{N}}$ along $\widehat{N}$. Thus critical points $X\colon B \to \mathbb{R}^{n+2}$ of $\mathscr{A}[X]$ necessarily satisfy

$$H \equiv 0 \quad \text{in } B.$$

Proof. Let $(u,v) \in B$ be conformal parameters. Choose an ONF $N = (N_1,\ldots,N_n)$ and consider normal variations

$$\widetilde{X}(u,v) = X(u,v) + \varepsilon\varphi(u,v)N_{\widehat{\gamma}}(u,v)$$

with arbitrary $\varphi \in C_0^\infty(B,\mathbb{R})$ and $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, where we set

$$N_{\widehat{\gamma}}(u,v) := \sum_{\sigma=1}^n \widehat{\gamma}^\sigma(u,v) N_\sigma(u,v), \quad \sum_{\sigma=1}^n \widehat{\gamma}^\sigma(u,v)^2 = 1.$$

We compute

$$\begin{aligned} \widetilde{X}_u &= X_u + \varepsilon\varphi_u N_{\widehat{\gamma}} + \varepsilon\varphi N_{\widehat{\gamma},u}\,, \\ \widetilde{X}_v &= X_v + \varepsilon\varphi_v N_{\widehat{\gamma}} + \varepsilon\varphi N_{\widehat{\gamma},v} \end{aligned}$$

as well as

$$\begin{aligned} \widetilde{X}_u^2 &= W + 2\varepsilon\varphi X_u \cdot N_{\widehat{\gamma},u} + \varepsilon^2\varphi_u^2 + \varepsilon^2\varphi^2 N_{\widehat{\gamma},u}^2\,, \\ \widetilde{X}_v^2 &= W + 2\varepsilon\varphi X_v \cdot N_{\widehat{\gamma},v} + \varepsilon^2\varphi_v^2 + \varepsilon^2\varphi^2 N_{\widehat{\gamma},v}^2\,, \\ \widetilde{X}_u \cdot \widetilde{X}_v &= \varepsilon\left\{X_u \cdot N_{\widehat{\gamma},v} + X_v \cdot N_{\widehat{\gamma},u}\right\}\varphi + \varepsilon^2\varphi_u\varphi_v + \varepsilon^2\varphi^2 N_{\widehat{\gamma},u} \cdot N_{\widehat{\gamma},v}\,. \end{aligned} \tag{9.1}$$

Let us now introduce the following auxiliary forms

$$L_{\widehat{\gamma},ij} := X_{u^iu^j} \cdot N_{\widehat{\gamma}} = -X_{u^i} \cdot N_{\widehat{\gamma},u^j} = -X_{u^j} \cdot N_{\widehat{\gamma},u^i}$$

taking

$$\begin{aligned} &N_{\widehat{\gamma},u^i} \cdot N_{\widehat{\gamma}} = 0 && (\text{because } N_{\widehat{\gamma}}^2 = 1) \\ &X_{u^i} \cdot N_{\widehat{\gamma},u^j} = -X_{u^iu^j} \cdot N_{\widehat{\gamma}} && (\text{because } X_{u^i} \cdot N_{\widehat{\gamma}} = 0) \end{aligned}$$

into account as well as $X_u \cdot N_{\widehat{\gamma},v} = X_v \cdot N_{\widehat{\gamma},u}$ due to the symmetrie of $L_{\sigma,ij}$. In particular, it holds

$$\begin{aligned} L_{\widehat{\gamma},ij} &= -X_{u^i} \cdot N_{\widehat{\gamma},u^j} = -X_{u^i} \cdot \left(\sum_{\sigma=1}^n \widehat{\gamma}^\sigma_{u^j} N_\sigma + \sum_{\sigma=1}^n \widehat{\gamma}^\sigma N_{\sigma,u^j} \right) \\ &= -\sum_{\sigma=1}^n \widehat{\gamma}^\sigma X_{u^i} \cdot N_{\sigma,u^j} = \sum_{\sigma=1}^n \widehat{\gamma}^\sigma L_{\sigma,ij} \end{aligned}$$

for $i,j = 1,2$ which shows $L_{\widehat{\gamma},ij} = L_{\widehat{\gamma},ji}$. We rewrite the equations (9.1) into the form

$$\begin{aligned} \widetilde{X}_u^2 &= W - 2\varepsilon\varphi L_{\widehat{\gamma},11} + \varepsilon^2\varphi_u^2 + \varepsilon^2\varphi^2 N_{\widehat{\gamma},u}^2\,, \\ \widetilde{X}_v^2 &= W - 2\varepsilon\varphi L_{\widehat{\gamma},22} + \varepsilon^2\varphi_v^2 + \varepsilon^2\varphi^2 N_{\widehat{\gamma},v}^2\,, \\ \widetilde{X}_u \cdot \widetilde{X}_v &= -2\varepsilon\varphi L_{\widehat{\gamma},12} + \varepsilon^2\varphi_u\varphi_v + \varepsilon^2\varphi^2 N_{\widehat{\gamma},u} \cdot N_{\widehat{\gamma},v} \end{aligned}$$

which implies

$$\delta g_{11} = -2\varphi L_{\widehat{\gamma},11}\,, \quad \delta g_{12} = -2\varphi L_{\widehat{\gamma},12}\,, \quad \delta g_{22} = -2\varphi L_{\widehat{\gamma},22}$$

with the settings

$$\delta g_{11} := \frac{\partial \widetilde{g}_{11}}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

etc. Now in view of

$$2W\,\delta W = g_{22}\,\delta g_{11} - 2g_{12}\,\delta g_{12} + g_{11}\,\delta g_{22}$$

with $g_{11} = g_{22}$ and $g_{12} \equiv 0$ we arrive at the first variation formula

$$\delta W = \frac{1}{2}\,\delta g_{11} + \frac{1}{2}\,\delta g_{22} = -\left\{L_{\widehat{\gamma},11} + L_{\widehat{\gamma},22}\right\}\varphi = -2H_{\widehat{\gamma}}W\varphi \tag{9.2}$$

where we set

$$H_{\widehat{\gamma}} := \frac{L_{\widehat{\gamma},11}g_{22} - L_{\widehat{\gamma},12}g_{12} + L_{\widehat{\gamma},22}g_{11}}{2W^2}\,.$$

This proves the statement. □

9.2 The first variation of non-parametric functionals. Minimal graphs

The general problem

In this section we want to consider surface graphs of the form

$$X(x,y) = (x,y,\zeta_1(x,y),\ldots,\zeta_n(x,y)), \quad \Omega \subset \mathbb{R}^2,$$

which are critical for the non-parametric area functional. We start with computing the first variation of a general non-parametric functional before we treat the special case of minimal surfaces.

We introduce the following abbreviations

$$p_\sigma := \frac{\partial \zeta_\sigma}{\partial x}, \quad q_\sigma := \frac{\partial \zeta_\sigma}{\partial y}, \quad \sigma = 1,\ldots,n,$$

as well as

$$\zeta = (\zeta_1,\ldots,\zeta_n), \quad p = (p_1,\ldots,p_n), \quad q = (q_1,\ldots,q_n),$$
$$r = (r_1,\ldots,r_{2n}) = (p_1,q_1,\ldots,p_n,q_n).$$

Let us now consider the general non-parametric functional

$$\mathscr{F}[\zeta] := \iint_\Omega F(x,y,\zeta,\nabla\zeta)\,dxdy = \iint_\Omega F(x,y,\zeta,r)\,dxdy.$$

For the next computations we refer the reader to Fomin and Gelfand [61], Funk [68] or Evans [53].

Theorem 9.2. *The system of n Euler-Lagrange equations for the functional* $\mathscr{F}[\zeta]$ *reads*

$$\frac{dF_{p_\sigma}(x,y,\zeta,r)}{dx}+\frac{dF_{q_\sigma}(x,y,\zeta,r)}{dy}=F_{z_\sigma}(x,y,\zeta,r)\quad \textit{for } \sigma=1,\ldots,n.$$

Proof. Let $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. We consider perturbations

$$\widetilde{\zeta}_\sigma=\zeta_\sigma+\varepsilon\varphi_\sigma$$

with $\varphi_\sigma\in C_0^\infty(\Omega,\mathbb{R})$ for $\sigma=1,\ldots,n$. Let $\varphi=(\varphi_1,\ldots,\varphi_n)$. Then it follows

$$\begin{aligned}
&\mathscr{F}[\widetilde{\zeta}]-\mathscr{F}[\zeta]\\
&\quad=\iint\limits_\Omega F(x,y,\zeta+\varepsilon\varphi,\nabla\zeta+\varepsilon\nabla\varphi)\,dxdy-\iint\limits_\Omega F(x,y,\zeta,\nabla\zeta)\,dxdy\\
&\quad=\varepsilon\sum_{\sigma=1}^n\iint\limits_\Omega F_{z_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_\sigma\,dxdy\\
&\qquad+\varepsilon\sum_{\sigma=1}^n\iint\limits_\Omega\Big\{F_{p_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_{\sigma,x}+F_{q_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_{\sigma,y}\Big\}\,dxdy+o(\varepsilon).
\end{aligned}$$

Notice that

$$\begin{aligned}
F_{p_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_{\sigma,x}&=\frac{d}{dx}\Big[F_{p_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_\sigma\Big]-\frac{dF_{p_\sigma}(x,y,\zeta,\nabla\zeta)}{dx}\,\varphi_\sigma\,,\\
F_{q_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_{\sigma,y}&=\frac{d}{dy}\Big[F_{q_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_\sigma\Big]-\frac{dF_{q_\sigma}(x,y,\zeta,\nabla\zeta)}{dy}\,\varphi_\sigma
\end{aligned}$$

as well as

$$\iint\limits_\Omega \operatorname{div}\big(F_{p_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_\sigma,F_{q_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_\sigma\big)\,dxdy=0\quad\text{for all }\sigma=1,\ldots,n$$

due to the choice of the test functions φ_σ. Thus we arrive at

$$\begin{aligned}
\mathscr{F}[\widetilde{\zeta}]-\mathscr{F}[\zeta]&=\varepsilon\sum_{\sigma=1}^n\iint\limits_\Omega F_{z_\sigma}(x,y,\zeta,\nabla\zeta)\varphi_\sigma\,dxdy\\
&\quad-\varepsilon\sum_{\sigma=1}^n\iint\limits_\Omega\left\{\frac{dF_{p_\sigma}(x,y,\zeta,\nabla\zeta)}{dx}+\frac{dF_{q_\sigma}(x,y,\zeta,\nabla\zeta)}{dy}\right\}\varphi_\sigma\,dxdy+o(\varepsilon).
\end{aligned}$$

The fundamental lemma of the calculus of variations proves the statement. □

Minimal graphs

We want to specify the foregoing considerations to the problem of establishing the Euler-Lagrange equations for area functional. Let $X(x,y)$ be a surface graph with tangential vectors

$$X_x = (1,0,\zeta_{1,x},\ldots,\zeta_{n,x}), \quad X_y = (0,1,\zeta_{1,y},\ldots,\zeta_{n,y}).$$

For the coefficients g_{ij} of its first fundamental form we therefore compute

$$g_{11} = 1 + \sum_{\sigma=1}^{n} \zeta_{\sigma,x}^2\,, \quad g_{12} = \sum_{\sigma=1}^{n} \zeta_{\sigma,x}\zeta_{\sigma,y}\,, \quad g_{22} = 1 + \sum_{\sigma=1}^{n} \zeta_{\sigma,y}^2\,,$$

and the squared area element $W(x,y)$ takes the form

$$\begin{aligned} W^2 &= g_{11}g_{22} - g_{12}^2 \\ &= 1 + \sum_{\sigma=1}^{n} \zeta_{\sigma,x}^2 + \sum_{\sigma=1}^{n} \zeta_{\sigma,y}^2 + \sum_{\sigma=1}^{n} \zeta_{\sigma,x}^2 \sum_{\sigma=1}^{n} \zeta_{\sigma,y}^2 - \left(\sum_{\sigma=1}^{n} \zeta_{\sigma,x}\zeta_{\sigma,y} \right)^2 . \end{aligned}$$

Thus the non-parametric area functional reads

$$\mathscr{A}[\zeta] = \iint_{\Omega} \sqrt{1 + \sum_{\sigma=1}^{n} \zeta_{\sigma,x}^2 + \sum_{\sigma=1}^{n} \zeta_{\sigma,y}^2 + \sum_{\sigma=1}^{n} \zeta_{\sigma,x}^2 \sum_{\sigma=1}^{n} \zeta_{\sigma,y}^2 - \left(\sum_{\sigma=1}^{n} \zeta_{\sigma,x}\zeta_{\sigma,y} \right)^2}\, dxdy. \tag{9.3}$$

In the special case $n = 1$ of one codimension there is only one function $\zeta(x,y)$ generating the surface graph, and the area functional can be written in the simpler form

$$\mathscr{A}[\zeta] = \iint_{\Omega} \sqrt{1 + \zeta_x^2 + \zeta_y^2}\, dxdy.$$

The system of n Euler-Lagrange equations then reduces to one single equation.

Now back to the case of arbitrary codimension. Using our abbreviations $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^n$ from above we infer

$$W \equiv F(p,q) = \sqrt{1 + p^2 + q^2 + p^2q^2 - (p \cdot q)^2}$$

for the integrand of the functional $\mathscr{A}[\zeta]$. Differentiation gives

$$\frac{\partial F(p,q)}{\partial p_\sigma} = \frac{p_\sigma + p_\sigma q^2 - q_\sigma (p \cdot q)}{W}\,, \quad \frac{\partial F(p,q)}{\partial q_\sigma} = \frac{q_\sigma + q_\sigma p^2 - p_\sigma (p \cdot q)}{W}\,.$$

Note furthermore $F_{z_\sigma}(p,q) \equiv 0$ for $\sigma = 1,\ldots,n$.

Corollary 9.1. *The non-parametric minimal surface system reads*

$$\operatorname{div}\frac{\nabla\zeta_\sigma}{W} = -\operatorname{div}\frac{\big(\zeta_{\sigma,x}q^2-\zeta_{\sigma,y}\,p\cdot q\,,\,\zeta_{\sigma,y}\,p^2-\zeta_{\sigma,x}\,p\cdot q\big)}{W} \quad \text{in } \Omega \tag{9.4}$$

for $\sigma = 1,\ldots,n$.

Note that the right hand side of this Euler-Lagrange system vanishes identically in case $n=1$ of one codimension. Then we particularly get

$$\operatorname{div}\frac{\nabla\zeta}{W} = 0 \quad \text{in } \Omega. \tag{9.5}$$

Let us finally note down (9.4) for graphs $X(x,y) = (x,y,\varphi(x,y),\psi(x,y))$:

$$\begin{aligned}\operatorname{div}\frac{\nabla\varphi}{W} &= -\operatorname{div}\frac{\big((\varphi_x\psi_y-\varphi_y\psi_x)\psi_y,(\varphi_y\psi_x-\varphi_x\psi_y)\psi_x\big)}{W}\,,\\ \operatorname{div}\frac{\nabla\psi}{W} &= -\operatorname{div}\frac{\big((\varphi_y\psi_x-\varphi_x\psi_y)\varphi_y,(\varphi_x\psi_y-\varphi_y\psi_x)\varphi_x\big)}{W}\,.\end{aligned}$$

9.3 Geometry of minimal surfaces

Following section 2.3, the characteristic property $H\equiv 0$ for a conformally parametrized minimal immersion $X\colon B\to\mathbb{R}^{n+2}$ is equivalent to

$$\Delta X(u,v) = 0 \quad \text{in } B.$$

From this Laplace equation we will derive some interesting geometric properties of minimal surfaces.

Embedding into subspaces

A first remarkable consequence is the following:

Let a minimal surface $X\colon B\to\mathbb{R}^4$ with $X(\partial B)\subset\mathbb{R}^3$ be given. Then the whole surface $X(B)$ is also contained in the subspace $\mathbb{R}^3$.

The general situation is the contents of the

Proposition 9.1. *Let the minimal surface $X\colon B\to\mathbb{R}^{n+2}$ satisfy*

$$X_u(u,v)\cdot e = X_v(u,v)\cdot e = 0 \quad \text{on } \partial B$$

with a fixed vector $e\in\mathbb{R}^{n+2}$. Then there hold

$$X_u(u,v)\cdot e = X_v(u,v)\cdot e = 0 \quad \text{in } B.$$

Proof. Introduce conformal parameters $(u,v) \in B$, and let $\Phi_i := X_{u^i} \cdot e$ for $i = 1,2$. Then we compute

$$\Delta \Phi_i = (X_{u^i uu} + X_{u^i vv}) \cdot e = (X_{uu} + X_{vv})_{u^i} \cdot e = (\Delta X \cdot e)_{u^i} = 0.$$

Thus the Φ_i are harmonic functions, and the maximum principle proves the statement. □

The convex hull property

Next let a minimal surface $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Then $H \equiv 0$ immediately implies

$$\kappa_{\sigma,1} + \kappa_{\sigma,2} \equiv 0$$

for the principal curvatures $\kappa_{\sigma,1}$ and $\kappa_{\sigma,2}$ along an unit normal vectors N_σ of N. Thus we obtain

$$K_\sigma = \kappa_{\sigma,1}\kappa_{\sigma,2} \le 0$$

for the Gauss curvatures along N_σ, and we infer

$$K = \sum_{\sigma=1}^{n} K_\sigma \le 0 \quad \text{in } B$$

for the Gauss curvature of the minimal immersion.

But surfaces with non-positive curvature satisfy the so-called convex hull property. In particular, using an idea which goes back to S. Hildebrandt we prove the

Proposition 9.2. *Let the minimal surface $X\colon B \to \mathbb{R}^{n+2}$ be given. Then it holds*

$$X(B) \subset \operatorname{conv} X(\partial B)$$

with $\operatorname{conv} X(\partial B)$ *denoting the convex hull of the curve* $X(\partial B)$.

Proof. Use conformal parameters $(u,v) \in B$. Choose a unit vector $Z \in \mathbb{R}^{n+2}$ with $|Z| = 1$ and a real number $h \in \mathbb{R}$ such that

$$X(\partial B) \subset \mathscr{H}_{Z,h} := \left\{ X \in \mathbb{R}^{n+2} : Z \cdot X - h \le 0 \right\}.$$

Along with the minimal immersion $X(u,v)$ we now define the function

$$\Phi(u,v) := Z \cdot X(u,v) - h, \quad (u,v) \in B,$$

which satisfies

$$\Delta \Phi(u,v) = 0 \quad \text{in } B, \quad \Phi(u,v) \le 0 \quad \text{on } \partial B.$$

Thus the maximum principle implies $\Phi(u,v) \leq 0$ in B, therefore $X(B) \subset \mathscr{H}_{Z,h}$. But on the other it holds

$$\operatorname{conv} X(\partial B) = \bigcap_{Z,h} \mathscr{H}_{Z,h}\,,$$

i.e. it follows $X(B) \subset \mathscr{H}_{Z,h}$ for all half spaces $\mathscr{H}_{Z,h}$ proving the statement. □

Various geometric enclosure theorems for minimal surfaces can be found e.g. in the monograph Burago and Zalgaller [20]. We also want to refer to Dierkes et al. [44], Nitsche [126], and Osserman [130]. For further developments we refer to Dierkes [42], [43].

In particular, enclosure theorems of this kind are useful for establishing curvature estimates as we will elaborate shortly.

9.4 A priori estimates for elliptic systems with quadratic growth

For this purpose we need some additional analytical facts from the theory of *quasilinear elliptic systems of second order with quadratic growth in the gradient.* We refer the reader to the work of Heinz [80] as well as to Schulz [146], Dierkes et al. [44], and Sauvigny [143].

Such elliptic systems played an important role for our considerations from chapter 8. In particular, the model problem

$$\Delta\Phi(u,v) = r(u,v) \quad \text{in } B$$

with the right hand side

$$r = \frac{\partial a}{\partial u}\frac{\partial b}{\partial v} - \frac{\partial a}{\partial v}\frac{\partial b}{\partial u}$$

discussed there *growth quadratically in the gradients of* a and b, i.e.

$$|\Delta\Phi| \leq \frac{1}{2}\,(|a_u|^2 + |b_v|^2 + |a_v|^2 + |b_v|^2) = \frac{1}{2}\,(|\nabla a|^2 + |\nabla b|^2).$$

In the following we consider mappings $X\colon B \to \mathbb{R}^{n+2}$ satisfying

$$|\Delta X| \leq \lambda\,|\nabla X|^2 + \mu \quad \text{in } B \tag{9.6}$$

with non-negative real $\lambda, \mu \in [0,\infty)$.

The following results are taken from Sauvigny [143], chapter 12.

Interior gradient estimates

Assume that the smallness condition

$$\lambda \cdot \sup_{(u,v)\in B} |X(u,v)| < 1 \quad \text{in } B$$

holds true. Let

$$\delta(w) := \operatorname{dist}(w, \partial B), \quad d := \sup_{w\in B} \delta(w).$$

Then there exists a constant $c_1 = c_1(\lambda,\mu,M,d) \in [0,\infty)$ *such that*

$$\delta(w)|\nabla X(w)| \le c_1(\lambda,\mu,M,d), \quad M := \sup_{(u,v)\in B} |X(u,v)|.$$

Likewise it holds

$$\|X\|_{C^{1+\alpha}(K_\varepsilon)} \le c_2(\lambda,\mu,M,d,\varepsilon,\alpha) \in [0,\infty) \quad \textit{for } \alpha \in (0,1)$$

for all compact domains $K_\varepsilon \subset\subset \mathring{B}$.

Global gradient estimates

If we otherwise assume additionally

$$X(u,v) = 0 \quad \text{on } \partial B$$

we can made a statement about a gradient estimate up to ∂B. *Namely it holds*

$$\|X\|_{C^{1+\alpha}(B)} \le c_3(\lambda,\mu,M,\alpha) \quad \textit{for } \alpha \in (0,1)$$

with a real constant $c_3 \in [0,\infty)$. The proof uses a reflection principle for (9.6).

Interior estimates for the Jacobian

Now consider a *plane mapping* $X\colon B \to B$ solving system (9.6) with $\lambda \in [0,\frac{1}{2})$, $X\colon \partial B \to \partial B$ topologically and positively oriented, $X(0,0) = (0,0)$, and

$$J_X(u,v) = \frac{\partial(x,y)}{\partial(u,v)} > 0 \quad \text{in } B$$

for its Jacobian. *Then it holds*

$$c_4(\lambda,\mu,r) \le |\nabla X(w)| \le c_5(\lambda,\mu,r) \quad \textit{for all compact } B_r \subset\subset \mathring{B}$$

with positive constants $c_4, c_5 \in (0,\infty)$.

If additionally

$$\iint_B (X_u^2 + X_v^2)\, dudv \le N$$

with a real constant $N \in (0,\infty)$, *then the latter estimate is true for* $0 \le \lambda < 1$, *i.e.*

$$c_4^*(\lambda,\mu,r,N) \le |\nabla X(w)| \le c_5^*(\lambda,\mu,r,N) \quad \textit{for all compact } B_r \subset\subset \mathring{B}$$

with constants $c_4^*, c_5^* \in (0,\infty)$.

9.5 A curvature estimate for minimal graphs with subquadratic growth

Using the analytical tools for nonlinear elliptic systems just presented together with a classical Schauder estimate from Gilbarg and Trudinger [71] we can proof our first curvature estimate.

Theorem 9.3. *Let the minimal graph*

$$X(x,y) = (x,y,\zeta_1(x,y),\ldots,\zeta_n(x,y))$$

on the disc closed disc $B_R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R\}$ *be given. Let*

$$X = X(u,v) \colon B \longrightarrow \mathbb{R}^{n+2}$$

be a reparametrization of this graph such that $X(u,v)$ *is conformally parametrized. Then there exists a real constant* $\Theta \in [0,+\infty)$ *such that for the Gaussian curvature* $K_N(0,0)$ *of* $X(u,v)$ *along an arbitrary unit normal vector* N *it holds*

$$|K_N(0,0)| \le \frac{\Theta}{R^4}\, \|X\|^2_{C^0(B_R)}\,.$$

Proof. 1. Consider the conformal representation $X(u,v)$ of the graph. Then we can ensure that

$$f(u,v) := (x(u,v), y(u,v)), \quad (u,v) \in B,$$

maps the boundary ∂B topologically and positively orientied onto ∂B_R, and it satisfies $f(0,0) = (0,0)$ as well as $J_f(w) > 0$ in B for the Jacobian of $f(u,v)$ since $X(u,v)$ represents a surface graph; see Sauvigny [143]. Due to

$$\triangle f(u,v) = 0 \quad \text{in } B$$

there exists a real constant $C_1 \in (0,+\infty)$ such that

$$|\nabla f(0,0)| \ge C_1 R.$$

We infer

$$W(0,0) = \frac{1}{2}|\nabla X(0,0)|^2 \geq \frac{1}{2}|\nabla f(0,0)|^2 \geq \frac{1}{2}C_1^2R^2\,.$$

2. Using conformal parameters it holds

$$\begin{aligned}|K_N(0,0)| &\leq \frac{|N(0,0)\cdot X_{uu}(0,0)||N\cdot X_{vv}(0,0)|+|N\cdot X_{uv}(0,0)|^2}{W(0,0)^2}\\ &\leq \frac{|X_{uu}(0,0)||X_{vv}(0,0)|+|X_{uv}(0,0)|^2}{W(0,0)^2}\,.\end{aligned}$$

Due to $\triangle X(u,v)=0$ in B there is a universal constant $C_2\in(0,+\infty)$ with the property (see e.g. Gilbarg and Trudinger [71], Theorem 4.6)

$$|X_{u^iu^j}(0,0)| \leq C_2\|X\|_{C^0(B_R)} \quad \text{for } i,j=1,2. \tag{9.7}$$

Note here that the supremum norm does not depend on the choice of the parameters. Putting all estimates together we arrive at

$$|K_N(0,0)| \leq \frac{8C_2^2}{C_1^4R^4}\|X\|^2_{C^0(B_R)} =: \frac{\Theta}{R^4}\|X\|^2_{C^0(B_R)}$$

setting $\Theta := 8C_2^2C_1^{-4}$. The statement is proved. □

9.6 A theorem of Bernstein-type

Complete geodesic discs

For our curvature estimates later we need some new terminology the reader can find in any textbook on Riemannian geometry or geometric analysis. Let us refer e.g. to Klingenberg [106] or Sauvigny [140].

We consider twodimensional simply connected and complete immersions M^2 in the following sense.

Definition 9.1. The regular surface $M^2\subset\mathbb{R}^{n+2}$ is called *complete* if for every point $p\in M^2$ any geodesic curve $\gamma\colon[0,\varepsilon)\to M^2$ with $\gamma(0)=p$ can be extended to a parametric geodesic curve $\widetilde{\gamma}\colon\mathbb{R}\to M^2$ defined on the whole real line $\mathbb{R}$.

Elementary examples of complete surfaces are the plane $\mathbb{R}^2$, the cylinder, or the sphere. The minimal surface (w,w^2) is also complete.

Geodesic curves are defined by means of the so-called *exponential map*

$$\exp_p\colon \mathfrak{T}_{M^2}(p)\longrightarrow M^2$$

on the twodimensional tangential space $\mathfrak{T}_{M^2}(p)$ of M^2 at $p\in M^2$.

For sufficiently small $r > 0$ satisfying

$$r\sqrt{K_0} < \pi$$

with a real constant K_0 such that

$$K(p) \leq K_0 \quad \text{for all } p \in M^2\,,$$

the exponential map represents a local diffeomorphism of a closed disc $B_r(p)$ of radius $r > 0$ with center $p \in M^2$ into the surface $M^2 \subset \mathbb{R}^{n+2}$, i.e. it maps one-to-one onto the *geodesic disc*

$$\mathscr{B}_r(p) := \exp_p(B_r(p)).$$

In particular, if it holds $K \leq 0$ for the Gaussian curvature of $M^2 \subset \mathbb{R}^{n+2}$ then the exponental map is injective for all $r \in (0,+\infty)$ by a theorem of Hadamard.

A Bernstein-type result

The following corollary generalizes a well-known result of Liouville which states *that every entire and bounded holomorphic functions must be constant.*

Corollary 9.2. *Let $X : \mathbb{R}^2 \to \mathbb{R}^{n+2}$ be a minimal graph of class $C^2(\mathbb{R}^2,\mathbb{R})$ satisfying the growth condition*

$$\|X\|_{C^0(B_R)} \leq \Omega_* R^\varepsilon\,, \quad \varepsilon \in (0,2), \tag{9.8}$$

for all $B_R \subset \mathbb{R}^2$ with a real constant $\Omega_* \in (0,+\infty)$. Then it holds

$$\lim_{R\to\infty} |K_N(0,0)| = 0 \quad \textit{for all unit normal vectors } N,$$

i.e. a complete minimal graph defined on the whole plane $\mathbb{R}^2$ represents a plane.

Proof. Since the minimal graph is defined over the whole plane $\mathbb{R}^2$ we may consider discs B_R with radius $R > 0$ arbitrarily large due to Hadamard's theorem mentioned above. The previous curvature estimate then implies

$$|K_N(0,0)| \leq \frac{\Theta\Omega_*^2}{R^4} R^{2\varepsilon} \longrightarrow 0 \quad \text{for } R \to \infty$$

for all unit normal vectors N proving the statement.

Remark 9.1.

1. This result is sharp in the following sense: *The mapping (w,w^2) for $w \in \mathbb{C}$ does not satisfy the assumed growth condition,* and obviously it is not a plane.
2. Since holomorphic mappings $(x,y) \mapsto (\varphi(x,y),\psi(x,y))$ generate minimal graphs (x,y,φ,ψ) in $\mathbb{R}^4$ we may considerer our corollary as a generalisation of Liouville's theorem.

We want to point out that *it is not necessary to suppose the growth condition (9.8) in case* $n = 1$ *of one codimension.* Rather it holds the

Proposition 9.3. *Ever minimal graph* $X \in C^2(R^2, \mathbb{R}^3)$ *represents a plane.*

The nonlinearity of the minimal surface equation

$$(1+\zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1+\zeta_x^2)\zeta_{yy} = 0$$

in case $n = 1$ allows us to skip such growth or boundness conditions. But minimal graphs in $\mathbb{R}^{n+2}$ are solutions of *systems of quasilinear elliptic equations* forcing us to additional assumptions.

For example, Kawai in [105] brings *stability* into play which will be our subject in the next chapters.

Proposition 9.4. *Given the complete and stable minimal graph* $X\colon \mathbb{R}^2 \to \mathbb{R}^4$. *Then either it is a plane or it is a graph generated by a holomorphic or anti-holomorphic function.*

9.7 Osserman's curvature estimate for minimal surfaces

In this section we want to present Osserman's curvature estimate for minimal surfaces $X\colon B \to \mathbb{R}^{n+2}$ from [127].

Theorem 9.4. *Let* $X\colon B \to \mathbb{R}^{n+2}$ *be a conformally parametrized minimal surface representing a geodesic discs* $\mathfrak{B}_r(X_0)$ *of radius* $r > 0$ *and with center* $X_0 := X(0,0)$. *Assume that the unit normal vectors at each point of the immersion make an angle* $\omega > 0$ *with some fixed direction in space. Then it holds the curvature estimate*

$$|K(0,0)| \le \frac{1}{r^2}\,\frac{32(n+1)}{\sin^4\omega}\,.$$

An estimate for the complex valued surface vector

Let the conformally parametrized minimal immersion $X\colon B \to \mathbb{R}^{n+2}$ be given. Set

$$\varphi_k := x_u^k - ix_v^k, \quad k = 1, \ldots, n+2,$$

and $\varphi = (\varphi_1, \ldots, \varphi_{n+2}) \in \mathbb{C}^{n+2}$. In particular, it holds

$$|\varphi|^2 = \sum_{k=1}^{n+2} |\varphi_k|^2 = \sum_{k=1}^{n+2} x_u^k + \sum_{k=1}^{n+2} x_v^k = X_u^2 + X_v^2 = 2W$$

with the area element $W > 0$ of the surface.

The following two results and their proofs are taken from Osserman [127], Lemma 1.1 and Lemma 1.2. The first lemma was formulated there for conformally parametrized minimal surfaces, but it also holds true for all immersions.

Lemma 9.1. *Let the conformally parametrized immersion* $X\colon B \to \mathbb{R}^{n+2}$ *be given. Then every normal vector* $N(w)$ *at* $w \in B$ *makes an angle of at least* $\omega > 0$ *with the* x_n*-axis if and only if*

$$\frac{|\varphi_{n+2}(w)|^2}{|\varphi(w)|^2} \geq \frac{1}{2}\sin^2\omega.$$

Proof. Let $e_{n+2} = (0,\ldots,0,1)$ be the unit vector in the positive x_{n+2}-direction. The condition that an arbitrary unit vector $Z = (z^1,\ldots,z^{n+2}) \in \mathbb{R}^{n+2}$ makes an angle of at least $\omega > 0$ with the x_{n+2}-axis is

$$|z^{n+2}| \leq \cos\omega.$$

Assume that e_{n+2} lies in the tangential plane at $X(w)$. Then we set $\omega = \frac{\pi}{2}$. Thus suppose in the following that e_{n+2} does not lie in the tangential plane at the point $X(w)$. Choose an orthonormal frame $\{V_1,V_2,V_3\}$ as a basis of the three-dimensional space $\operatorname{Span}\{X_u,X_v,e_{n+2}\}$ such that

$$V_3 := e_{n+2}\,.$$

Let furthermore $Z \in \operatorname{span}\{V_1,V_2,V_3\}$ be a unit vector with the property $Z \perp X_1,X_2$ where we set

$$X_1 := \frac{X_u}{\sqrt{W}}\,, \quad X_2 := \frac{X_v}{\sqrt{W}}\,, \quad X_3 := Z.$$

Now we define coefficients v_{ik} by

$$X_1 = \sum_{k=1}^{3} v_{1k}V_k\,, \quad X_2 = \sum_{k=1}^{3} v_{2k}V_k\,, \quad X_3 = \sum_{k=1}^{3} v_{3k}V_k\,.$$

Then $(v_{ik})_{i,k=1,2,3} \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix with $v_{13}^2 + v_{23}^2 + v_{33}^2 = 1$ since $\{X_1,X_2,X_3\}$ and $\{V_1,V_2,V_3\}$ are orthonormal frames. But on the other hand

$$v_{i3} = X_i \cdot V_3 = X_i \cdot e_{n+2} = x_i^{n+2}\,,$$

and therefore

$$|\varphi_{n+2}|^2 = (x_u^{n+2})^2 + (x_v^{n+2})^2 = \big[(X_1\cdot V_3)^2 + (X_2\cdot V_3)^2\big]W.$$

Thus we infer

$$\frac{|\varphi_{n+2}|^2}{|\varphi|^2} = \frac{1}{2}\,(v_{13}^2 + v_{23}^2) = \frac{1}{2}\,(1 - v_{33}^2).$$

But $v_{33} = z^3 = \cos\vartheta$ with ϑ being the angle between the unit normal vector Z and the e_n-axis.

Now if all normals make an angle of at least ω then $|v_{33}| \le \cos\omega$, and it follows

$$\frac{|\varphi_{n+2}|^2}{|\varphi|^2} \ge \frac{1}{2}(1-\cos^2\omega) = \frac{1}{2}\sin^2\omega.$$

Conversely, if the inequality from the lemma holds true we find $|v_{33}| \le \cos\omega$. We extend $\{X_1,X_2,X_3\}$ to an orthonormal basis $\{X_1,\ldots,X_{n+2}\}$ of $\mathbb{R}^{n+2}$ such that

$$N = \lambda_3 X_3 + \ldots + \lambda_{n+2}X_{n+2}$$

for an arbitrary unit normal vector N of the surface with suitable coefficients $\lambda_3,\ldots,\lambda_{n+2}$. We infer $X_k \cdot e_{n+2} = 0$ for all $k = 4,\ldots,n+2$. Hence the angle ϑ between N and e_{n+2} satisfies

$$\cos\vartheta = N \cdot e_{n+2} = \lambda_3 X_3 \cdot e_{n+2} = \lambda_3 \cos\omega \le \cos\omega$$

since $\lambda_3 = N \cdot X_3 \le 1$. The lemma is proved. $\square$

Lemma 9.2. *Let the conformally parametrized minimal surface $X\colon B \to \mathbb{R}^{n+2}$ represent a geodesic disc of radius $r > 0$ with center $X_0 = X(0,0)$. Assume that every normal vector makes an angle of at least $\omega > 0$ with the x_{n+2}-axis. Then it holds*

$$|\varphi_{n+2}(0,0)| \ge r\sin\omega.$$

Proof. Denote by $\Gamma(\overline{B})$ the set of all continuously differentiable curves γ in B connecting the origin $(0,0) \in B$ with the boundary ∂B.

1. Now consider a smooth embedded curve

$$c^*(t) = \big(u(t),v(t)\big)\colon I \subset \mathbb{R} \longrightarrow B$$

from the set $\Gamma(\overline{B})$ with $c(0) = (0,0)$. Its length on the surface is given by

$$\mathscr{L}[c^*] = \int\limits_{c^*} \sqrt{W(\zeta)}\,d\zeta$$

with the positive line element $d\zeta = \sqrt{\dot u^2 + \dot v^2}\,dt$. Using the complex-valued functions φ and φ_{n+2} from the previous lemma we infer

$$\int\limits_{c^*} \sqrt{W(\zeta)}\,d\zeta = \frac{1}{\sqrt{2}} \int\limits_{c^*} |\varphi(\zeta)|\,d\zeta \le \frac{1}{\sin\omega} \int\limits_{c^*} |\varphi_{n+2}(\zeta)|\,d\zeta.$$

2. We want to estimate $|\varphi_{n+2}(\zeta)|$. For this purpose consider the complex-valued integral

$$F(w) := \int\limits_0^w \varphi_{n+2}(\zeta)\,d\zeta, \quad w \in B.$$

Since $\varphi_{n+2}(\zeta) \neq (0,0)$ by the foregoing lemma, the complex-valued derivative $F'(w)$ does not vanish:

$$|F'(w)| \neq 0 \quad \text{in } B, \quad \text{where } F'(w) = \partial_w F(w) = \frac{1}{2}\big[F_u(w) - iF_v(w)\big].$$

Thus $\sigma = F(w)$ has an inverse $w = G(\sigma)$ in some circle about $\sigma = (0,0)$. If $R > 0$ denotes the radius of the largest of such circles, we find a point $\sigma_0 \in \mathbb{C}$ with $|\sigma_0| = R$ such that $G(\sigma)$ can not be extended to a neighborhood of σ_0. Now let γ denote the line segment connecting $\sigma = (0,0)$ and σ_0, and consider its image c^* under the mapping G. In particular, c^* is a curve going from $w = (0,0)$ to ∂B, and $|\sigma| < R$ is mapped into $|w| < 1$. Finally it holds $G(0,0) = (0,0)$. Thus Schwarz' lemma[1] yields

$$|G'(0,0)| \leq \frac{1}{R}.$$

We infer

$$\begin{aligned} r &\leq \inf_{c \in \Gamma(\overline{B})} \int_c \sqrt{W}\, d\zeta \leq \int_{c^*} \sqrt{W}\, d\zeta \leq \frac{1}{\sin\omega} \int_{c^*} |\varphi_{n+1}|\, d\zeta \leq \frac{1}{\sin\omega} \int_\gamma d\sigma \\ &= \frac{R}{\sin\omega} \leq \frac{1}{|G'(0,0)|\sin\omega} = \frac{|F'(0,0)|}{\sin\omega} = \frac{|\varphi_{n+2}(0,0)|}{\sin\omega} \end{aligned}$$

from where the statement follows. $\Box$

A representation of the Gauss curvature

The following representation of the Gaussian curvature K of a conformally parametrized minimal surface using the complex-valued functions φ_k from above goes also back to Osserman [127].

Lemma 9.3. *Let the conformally parametrized minimal surface $X\colon B \to \mathbb{R}^{n+2}$ be given. Then it holds*

$$-2KW^2 = \frac{1}{2W} \sum_{k,\ell=1}^{n+2} |\varphi_k \varphi_\ell' - \varphi_k' \varphi_\ell|^2$$

for its Gaussian curvature K.

Proof. From section 3.5 we recall the conformal representation

$$-2KW = \left(\frac{W_u}{W}\right)_u + \left(\frac{W_v}{W}\right)_v = \frac{1}{W}\,\Delta W - \frac{1}{W^2}\,(W_u^2 + W_v^2).$$

[1] Let the holomorphic function $\phi\colon B_R \to B$ with $\phi(0,0) = (0,0)$ be given. Then $|\phi'(0,0)| \leq \frac{1}{R}$.

Furthermore, the complex-valued vector $\varphi = (\varphi_1, \dots, \varphi_{n+2})$ with the holomorphic functions φ_k satisfies

$$|\varphi|^2 = \sum_{k=1}^{n+2} |\varphi_k|^2 = 2W.$$

We start with evaluating the Laplacian (let $\varphi' = \varphi_w$ and note $\varphi_{\overline{w}} \equiv 0$)

$$\begin{aligned}\frac{1}{4}\Delta W &= W_{w\overline{w}} = \frac{1}{2}\partial_{w\overline{w}}|\varphi|^2 = \frac{1}{2}\partial_w \sum_{k=1}^{n+2}\left\{\overline{\varphi}_k\varphi_{k,\overline{w}} + \varphi_k\overline{\varphi_{k,w}}\right\} \\ &= \frac{1}{2}\partial_w \sum_{k=1}^{n+2}\varphi_k\overline{\varphi_{k,w}} = \frac{1}{2}\partial_w \sum_{k=1}^{n+2}\varphi_k\overline{\varphi_k'} = \frac{1}{2}\sum_{k=1}^{n+2}\varphi_k'\overline{\varphi_k'}.\end{aligned}$$

Taking next

$$W_u^2 + W_v^2 = 4W_w W_{\overline{w}}$$

into account we compute

$$W_w = \frac{1}{2}\partial_w|\varphi|^2 = \frac{1}{2}\sum_{k=1}^{n+2}\left\{\varphi_{k,w}\overline{\varphi}_k + \varphi_k\overline{\varphi_{k,\overline{w}}}\right\} = \frac{1}{2}\sum_{k=1}^{n+2}\varphi_k'\overline{\varphi}_k\,,$$

and analogously

$$W_{\overline{w}} = \frac{1}{2}\sum_{k=1}^{n+2}\varphi_k\overline{\varphi_k'}.$$

It follows that

$$\begin{aligned}W_{w\overline{w}} - \frac{1}{W}W_w W_{\overline{w}} &= \frac{1}{2}\sum_{k=1}^{n+2}\varphi_k'\overline{\varphi_k'} - \frac{1}{4W}\sum_{k,\ell=1}^{n+2}\varphi_k'\overline{\varphi_k}\varphi_\ell\overline{\varphi_\ell'} \\ &= \frac{1}{2}\sum_{k=1}^{n+2}\varphi_k'\overline{\varphi_k'} - \frac{\sum_{k,\ell=1}^{n+2}\varphi_k'\overline{\varphi_k}\varphi_\ell\overline{\varphi_\ell'}}{2\sum_{\ell=1}^{n+2}\varphi_\ell\overline{\varphi_\ell}} \\ &= \frac{1}{2}\,\frac{\sum_{k,\ell=1}^{n+2}\left(\varphi_k'\overline{\varphi_k'}\varphi_\ell\overline{\varphi_\ell} - \varphi_k'\overline{\varphi_k}\varphi_\ell\overline{\varphi_\ell'}\right)}{\sum_{\ell=1}^{n+2}\varphi_\ell\overline{\varphi_\ell}}\,.\end{aligned}$$

On the other hand we calculate

$$\begin{aligned}|\varphi_k\varphi_\ell' - \varphi_k'\varphi_\ell|^2 &= (\varphi_k\varphi_\ell' - \varphi_k'\varphi_\ell)(\overline{\varphi_k}\overline{\varphi_\ell'} - \overline{\varphi_k'}\overline{\varphi_\ell}) \\ &= \varphi_k\overline{\varphi_k}\varphi_\ell'\overline{\varphi_\ell'} - \varphi_k\varphi_\ell'\overline{\varphi_k'}\overline{\varphi_\ell} - \overline{\varphi_k}\overline{\varphi_\ell'}\varphi_k'\varphi_\ell + \varphi_k'\varphi_\ell\overline{\varphi_k'}\overline{\varphi_\ell}\,.\end{aligned}$$

We thus infer

$$\begin{aligned}
&\sum_{k,\ell=1}^{n+2} |\varphi_k\varphi'_\ell - \varphi'_k\varphi_\ell|^2 \\
&\quad = \sum_{k,\ell=1}^{n+2}\Big\{\varphi_k\overline{\varphi_k}\varphi'_\ell\overline{\varphi'_\ell} + \varphi'_k\varphi_\ell\overline{\varphi'_k}\overline{\varphi_\ell}\Big\} - \sum_{k,\ell=1}^{n+2}\Big\{\varphi_k\varphi'_\ell\overline{\varphi'_k}\overline{\varphi_\ell} + \overline{\varphi_k}\overline{\varphi'_\ell}\varphi'_k\varphi_\ell\Big\} \\
&\quad = 2\sum_{k,\ell=1}^{n+2}\Big\{\varphi'_k\overline{\varphi'_k}\varphi_\ell\overline{\varphi_\ell} - \overline{\varphi_k}\varphi'_k\varphi_\ell\overline{\varphi'_\ell}\Big\}.
\end{aligned}$$

Altogether we have derived

$$\begin{aligned}
-2KW^2 &= \Delta W - \frac{|\nabla W|^2}{W} = 4W_{w\overline{w}} - \frac{4}{W}W_wW_{\overline{w}} = \frac{1}{|\varphi|^2}\sum_{k,\ell=1}^{n+2}|\varphi_k\varphi'_\ell - \varphi'_k\varphi_\ell|^2 \\
&= \frac{1}{2W}\sum_{k,\ell=1}^{n+2}|\varphi_k\varphi'_\ell - \varphi'_k\varphi_\ell|^2
\end{aligned}$$

which proves the statement. □

Now introduce the meromorphic function

$$f_k(w) := \frac{\varphi_k(w)}{\varphi_{n+2}(w)}, \quad k = 1, \ldots, n+1.$$

For its derivatives we compute

$$f'_k = \frac{\varphi'_k\varphi_{n+2} - \varphi_k\varphi'_{n+2}}{\varphi_{n+2}^2}, \quad k = 1, \ldots, n+1,$$

as well as

$$f_kf'_\ell - f'_kf_\ell = \frac{(\varphi'_\ell\varphi_{n+2} - \varphi_\ell\varphi'_{n+2})\varphi_k - (\varphi'_k\varphi_{n+2} - \varphi_k\varphi'_{n+2})\varphi_\ell}{\varphi_{n+2}^4} = \frac{\varphi'_\ell\varphi_k - \varphi'_k\varphi_\ell}{\varphi_{n+2}^3}$$

for $k, \ell = 1, \ldots, n+1$.

Lemma 9.4. *The Gaussian curvature K of the conformally parametrized minimal surface satisfies*

$$K = -\frac{2}{|\varphi_{n+2}|^2}\,\frac{\displaystyle\sum_{k,\ell=1}^{n+1}|f_kf'_\ell - f'_kf_\ell|^2 + 2\sum_{k=1}^{n+1}|f'_k|^2}{\left(1 + \displaystyle\sum_{k=1}^{n+1}|f_k|^2\right)^3}.$$

Proof. We compute

$$K = -\frac{1}{4W^3}\sum_{k,\ell=1}^{n+2}|\varphi_k\varphi'_\ell - \varphi'_k\varphi_\ell|^2 = -\frac{2}{\left(\sum\limits_{k=1}^{n+2}|\varphi_k|^2\right)^3}\sum_{k,\ell=1}^{n+2}|\varphi_k\varphi'_\ell - \varphi'_k\varphi_\ell|^2$$

$$= -2\,\frac{\sum\limits_{k,\ell=1}^{n+1}|\varphi_k\varphi'_\ell - \varphi'_k\varphi_\ell|^2 + 2\sum\limits_{k=1}^{n+1}|\varphi_k\varphi'_{n+2} - \varphi'_k\varphi_{n+2}|^2}{\left(|\varphi_{n+2}|^2 + \sum\limits_{k=1}^{n+1}|\varphi_k|^2\right)^3}$$

$$= -2|\varphi_{n+2}|^4\,\frac{\sum\limits_{k,\ell=1}^{n+1}|f_k f'_\ell - f'_k f_\ell|^2 + 2\sum\limits_{k=1}^{n+1}|f'_k|^2}{|\varphi_{n+2}|^6\left(1 + \sum\limits_{k=1}^{n+1}\frac{|\varphi_k|^2}{|\varphi_{n+2}|^2}\right)^3}$$

$$= -\frac{2}{|\varphi_{n+2}|^2}\,\frac{\sum\limits_{k,\ell=1}^{n+1}|f_k f'_\ell - f'_k f_\ell|^2 + 2\sum\limits_{k=1}^{n+1}|f'_k|^2}{\left(1 + \sum\limits_{k=1}^{n+1}|f_k|^2\right)^3}$$

This is the stated formula. □

Proof of Osserman's curvature estimate

Under the assumptions of Theorem 9.4 we infer from Lemma 9.1

$$|\varphi_{n+2}(w)|^2\sum_{k=1}^{n+1}|f_k(w)|^2 = \sum_{k=1}^{n+1}|\varphi_k(w)|^2 \le \sum_{k=1}^{n+2}|\varphi_k(w)|^2 = |\varphi(w)|^2 \le \frac{2|\varphi_{n+2}(w)|^2}{\sin^2\omega}$$

showing the uniform estimate

$$\sum_{k=1}^{n+1}|f_k(w)|^2 \le \frac{2}{\sin^2\omega} \quad \text{or} \quad \sum_{k=1}^{n+1}\|f_k\|^2_{C^0(B)} \le \frac{2}{\sin^2\omega}\,.$$

Now Schwarz' lemma implies

$$|f'_k(0,0)| \le \|f_k\|_{C^0(B)} - \frac{|f_k(0,0)|^2}{\|f_k\|_{C^0(B)}} =: \eta_k\|f_k\|_{C^0(B)}$$

with the abbreviation

$$\eta_k := 1 - \frac{|f_k(0,0)|^2}{\|f_k\|^2_{C^0(B)}}.$$

We estimate the Gauss curvature at the point $w = (0,0)$. For this purpose we use the Cauchy-Schwarz inequality to get

$$\begin{aligned}
&|f_k(0,0)f_\ell'(0,0) - f_k'(0,0)f_\ell(0,0)|^2 \\
&\quad \leq \big(|f_k(0,0)||f_\ell'(0,0)| + |f_k'(0,0)||f_\ell(0,0)|\big)^2 \\
&\quad \leq 2\big(|f_k(0,0)|^2||f_\ell'(0,0)|^2 + |f_k'(0,0)|^2|f_\ell(0,0)|^2\big)
\end{aligned}$$

which in turn gives

$$\begin{aligned}
&\sum_{k,\ell=1}^{n+1} |f_k(0,0)f_\ell'(0,0) - f_k'(0,0)f_\ell(0,0)|^2 \\
&\qquad \leq 4\sum_{k,\ell=1}^{n+1} |f_k(0,0)|^2|f_\ell(0,0)|^2 \leq 4\sum_{k=1}^{n+1}|f_k(0,0)|^2 \sum_{\ell=1}^{n+1}|f_\ell'(0,0)|^2.
\end{aligned}$$

Thus we infer

$$\begin{aligned}
&\sum_{k,\ell=1}^{n+1} |f_k(0,0)f_\ell'(0,0) - f_k'(0,0)f_\ell(0,0)|^2 + \sum_{\ell=1}^{n+1}|f_\ell'(0,0)|^2 \\
&\quad \leq 4\sum_{k=1}^{n+1}|f_k(0,0)|^2 \sum_{\ell=1}^{n+1}|f_\ell'(0,0)|^2 + \sum_{\ell=1}^{n+1}|f_\ell'(0,0)|^2 \\
&\quad = \left(4\sum_{k=1}^{n+1}|f_k(0,0)|^2 + 1\right)\sum_{\ell=1}^{n+1}|f_\ell'(0,0)|^2 \\
&\quad \leq \left(4\sum_{k=1}^{n+1}|f_k(0,0)|^2 + 1\right)\sum_{\ell=1}^{n+1}\eta_\ell^2\|f_\ell\|^2_{C^0(B)} \\
&\quad \leq 4\left(\sum_{k=1}^{n+1}|f_k(0,0)|^2 + 1\right)\sum_{\ell=1}^{n+1}\|f_\ell\|^2_{C^0(B)}\sum_{\ell=1}^{n+1}\eta_\ell^2 \\
&\quad \leq \frac{8}{\sin^2\omega}\sum_{\ell=1}^{n+1}\eta_\ell^2\left(\sum_{\ell=1}^{n+1}|f_\ell(0,0)|^2 + 1\right).
\end{aligned}$$

Inserting into the above repesentation for the Gaussian curvature yields

$$|K(0,0)| \leq -\frac{4}{|\varphi_{n+2}|^2} \frac{\sum_{k,\ell=1}^{n+1} |f_k f_\ell' - f_k' f_\ell|^2 + \sum_{k=1}^{n+1} |f_k|^2}{\left(1+\sum_{k=1}^{n+2}|f_k|^2\right)^3} \Big|_{w=(0,0)}$$

$$\leq \frac{32}{\sin^2\omega} \frac{\sum_{k=1}^{n+1}\eta_k^2}{|\varphi_{n+2}|^2\left(1+\sum_{k=1}^{n+1}|f_k|^2\right)^2} \Big|_{w=(0,0)}$$

$$\leq \frac{32(n+1)}{\sin^2\omega|\varphi_{n+2}(0,0)|^2}$$

since $\eta_k \in [0,1]$ for all $k = 1,\ldots,n+1$. But from Lemma 9.2 we know

$$|\varphi_{n+2}(0,0)|^2 \geq r^2 \sin^2\omega$$

with the geodesic radius $r > 0$ of the minimal disc. The finally proves the curvature estimate. □

9.8 A theorem of Bernstein type due to Osserman

From Osserman's curvature estimate we want to derive our next theorem of Bernstein type for twodimensional minimal surfaces in $\mathbb{R}^{n+2}$.

Corollary 9.3. *Let the minimal graph $X : \mathbb{R}^2 \to \mathbb{R}^{n+2}$ be given. Assume that its unit normal vectors at each point make an angle $\omega > 0$ with some fixed direction in space. Then X represents a plane.*

Proof. Due to Hadamard's theorem (see section 9.6) we may introduce geodesic minimal discs $\mathfrak{B}_r(X_0)$ for all $r > 0$ with $X_0 := X(0,0)$. The curvature estimate

$$|K(0,0)| \leq \frac{1}{r^2} \frac{32(n+1)}{\sin^4\omega}$$

implies

$$\lim_{r\to\infty} K(0,0) = 0.$$

But $X_0 = X(0,0)$ is chosen arbitrarily. The statement follows. □

9.9 Gradient estimates for minimal graphs

We conclude this chapter with elementary gradients estimates for minimal graphs. Such a-priori bounds are sufficient for solving the Dirichlet problem for this class of quasilinear equations. For we do not elaborate on this existence problem we want to refer the reader e.g. to Taylor [155] for detailed studies.

Gradient estimates on the boundary

Let us restrict to minimal graphs defined on the closed disc

$$B_R := \left\{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 \le R^2\right\}.$$

We want to start with establishing gradient estimates on the boundary.

Theorem 9.5. *Let $\zeta_\sigma \in C^2(\overline{B}_R,\mathbb{R})$, $\sigma = 1,\ldots,n$, be a solution of the minimal surface system*

$$\left(1+\sum_{\omega=1}^n \zeta_{\omega,y}^2\right)\zeta_{\sigma,xx} + 2\sum_{\omega=1}^n \zeta_{\omega,x}\zeta_{\omega,y}\zeta_{\sigma,xy} + \left(1+\sum_{\omega=1}^n \zeta_{\omega,x}^2\right)\zeta_{\sigma,yy} = 0 \quad \textit{in } B_R$$

for $\sigma = 1,\ldots,n$. Then there exists a real constant

$$C = C(\|\zeta_1\|_{C^2(\partial B_R)},\ldots,\|\zeta_n\|_{C^2(\partial B_R)},R)$$

such that

$$\max\left\{|\nabla\zeta_1(z)|,\ldots,|\nabla\zeta_n(z)|\right\} \le C(\|\zeta_1\|_{C^2(\partial B_R)},\ldots,\|\zeta_n\|_{C^2(\partial B_R)},R) \tag{9.9}$$

for all $z \in \partial B_R$.

Proof. Choose a point $z_0 = (x_0,y_0) \in \partial B_R$. For real parameters $c_1,c_2 \in (0,+\infty)$ consider the auxiliary functions

$$S_{\zeta_\sigma}^-(z) := c_1\log\left(1+c_2 d_\Gamma(z)\right) - \left\{\zeta_\sigma(z)-\zeta_\sigma(z_0)\right\}, \quad z=(x,y)\in B_R\,,$$

fo $\sigma = 1,\ldots,n$ where $d_\Gamma = d_\Gamma(z)$ means the distance of $z \in B_R$ to the tangential line $\Gamma \subset \mathbb{R}^2$ on ∂B_R at $z_0 \in \partial B_R$. We have

$$S_{\zeta_\sigma}^-(z_0) = 0 \quad \text{for all } \sigma = 1,\ldots,n. \tag{9.10}$$

Now we consider $\zeta := \zeta_1$. Let $C_\zeta \in [0,+\infty)$ be defined as

$$C_\zeta := \|\zeta\|_{C^2(\partial B_R)}\,. \tag{9.11}$$

Then we may choose $c_1 = c_1(C_\zeta, R)$ and $c_2 = c_2(C_\zeta, R)$ such that

$$S_\zeta^- > 0 \quad \text{for all } z \in \partial B_R \setminus \{z_0\}. \tag{9.12}$$

Let us now abbreviate

$$\mathscr{M}[\zeta_\omega] := a_{11}\zeta_{\omega,xx} + 2a_{12}\zeta_{\omega,xy} + a_{22}\zeta_{\omega,yy}\,, \quad \omega = 1,\ldots,n,$$

for the minimal surface operator with coefficients

$$a_{11} := 1 + \sum_{\sigma=1}^n \zeta_{\sigma,y}^2\,, \quad a_{12} := \sum_{\sigma=1}^n \zeta_{\sigma,x}\zeta_{\sigma,y}\,, \quad a_{22} := 1 + \sum_{\sigma=1}^n \zeta_{\sigma,x}^2\,.$$

There hold

$$\mathscr{M}[\zeta_\sigma] = 0 \quad \text{in } B_R$$

for $\sigma = 1,\ldots,n$. We compute the partial derivatives

$$S_{\zeta,x}^- = \frac{c_1 c_2}{1 + c_2 d_\Gamma}\, d_{\Gamma,x} - \zeta_x\,, \quad S_{\zeta,y}^- = \frac{c_1 c_2}{1 + c_2 d_\Gamma}\, d_{\Gamma,y} - \zeta_y$$

as well as

$$S_{\zeta,xx}^- = -\frac{c_1 c_2^2}{(1 + c_2 d_\Gamma)^2}\, d_{\Gamma,x}^2 - \zeta_{xx}\,,$$

$$S_{\zeta,xy}^- = -\frac{c_1 c_2^2}{(1 + c_2 d_\Gamma)^2}\, d_{\Gamma,x} d_{\Gamma,y} - \zeta_{xy}\,,$$

$$S_{\zeta,yy}^- = -\frac{c_1 c_2^2}{(1 + c_2 d_\Gamma)^2}\, d_{\Gamma,y}^2 - \zeta_{yy}\,.$$

Thus it follows

$$\begin{aligned}
\mathscr{M}[S_\zeta^-] &= a_{11} S_{\zeta,xx}^- + 2a_{12} S_{\zeta,xy}^- + a_{22} S_{\zeta,yy}^- \\
&= -\frac{c_1 c_2^2}{(1 + c_2 d_\Gamma)^2}\,(a_{11} d_{\Gamma,x}^2 + 2a_{12} d_{\Gamma,x} d_{\Gamma,y} + a_{22} d_{\Gamma,y}^2) - \mathscr{M}[\zeta] \\
&= -\frac{c_1 c_2^2}{(1 + c_2 d_\Gamma)^2}\,(a_{11} d_{\Gamma,x}^2 + 2a_{12} d_{\Gamma,x} d_{\Gamma,y} + a_{22} d_{\Gamma,y}^2).
\end{aligned}$$

Taking the ellipticity condition $a_{11}a_{22} - a_{12}^2 > 0$ together with $a_{11} > 0$ into account we conclude

$$\mathscr{M}[S_\zeta^-] \le 0 \quad \text{in } B_R\,.$$

Along with (9.10) and (9.12), the maximum principle yields $S_\zeta^-(z) \ge 0$ in B_R, i.e.

$$\zeta(z) - \zeta(z_0) \le c_1 \log\big(1 + c_2 d_\Gamma(z)\big) \quad \text{for all } z \in B_R\,. \tag{9.13}$$

Analogously we proceed with the function

$$S_\zeta^+(z) := c_1 \log\big(1 + c_2 d_\Gamma(z)\big) + \big\{\zeta(z) - \zeta(z_0)\big\}$$

and arrive at

$$\mathscr{M}[S_\zeta^+] \le 0 \quad \text{in } B_R\,, \quad S_\zeta^+(Z) \ge 0 \quad \text{for all } z \in \partial B_R\,,$$

i.e. $S_\zeta^+ \ge 0$ in B_R resp.

$$-c_1 \log\big(1 + c_2 d_\Gamma(z)\big) \le \zeta(z) - \zeta(z_0) \quad \text{for all } z \in B_R\,. \tag{9.14}$$

Taking the estimates (9.13) and (9.14) together gives

$$|\zeta(z) - \zeta(z_0)| \le c_1 \log\big(1 + c_2 d_\Gamma(z)\big) \quad \text{for all } z \in B_R\,.$$

Finally, for the outer unit normal vector ν at $z_0 \in \partial B$ we choose a point $z \in B_R$ such that

$$\left|\frac{\partial \zeta}{\partial \nu}(z_0)\right| \le \lim_{d_\Gamma(z)\to 0} \frac{|\zeta(z) - \zeta(z_0)|}{d_\Gamma(z)} \le \lim_{d_\Gamma(z)\to 0} \frac{c_1 \log\big(1 + c_2 d_\Gamma(z)\big)}{d_\Gamma(z)} = c_1 c_2\,.$$

Together with (9.11), the choice of c_1 and c_2, and finally a similar calculation for $\zeta_2, \zeta_3, \ldots, \zeta_n$ we get the statement. □

Barrier constructions for minimal immersions and surfaces with prescribed mean curvature as presented here can be found e.g. in Dierkes [43], Jorge and Tomi [100], or Bergner [11].

Global gradient estimates

Finally we discuss a global gradient estimate for minimal graphs on strictly convex domains of definition. A suitable three-point-condition which already goes back to Schauder [144] ensures that the minimal surface system decouples.

Theorem 9.6. *Let $\zeta_\sigma \in C^2(\mathring{B}_R, \mathbb{R}) \cap C^1(B_R, \mathbb{R})$, $\sigma = 1, \ldots, n$, be a minimal graph together with Dirichlet boundary conditions $\zeta_{R,\sigma} \in C^1(\partial B_R, \mathbb{R})$. Let*

$$C_\zeta^* := \max_{\sigma=1,\ldots,n} \|\zeta_{R,\sigma}\|_{C^1(\partial B_R)}\,.$$

Then there exists a real constant $C = C(C_\zeta^*, R) \in (0, +\infty)$ such that it holds

$$[\zeta_\sigma]_{C^1(\mathring{B}_R)} \le C(\zeta_\sigma, \mathbb{R})\,.$$

Sketch of the proof. Insert $(\zeta_1,\ldots,\zeta_n)$ and its gradient into the minimal surface system to get the decoupled system

$$\begin{aligned}
a_1(x,y)\zeta_{1,xx}+2b_1(x,y)\zeta_{1,xy}+c_1(x,y)\zeta_{1,yy} &= 0,\\
a_2(x,y)\zeta_{2,xx}+2b_2(x,y)\zeta_{2,xy}+c_2(x,y)\zeta_{2,yy} &= 0,\\
&\vdots\\
a_n(x,y)\zeta_{n,xx}+2b_n(x,y)\zeta_{n,xy}+c_n(x,y)\zeta_{n,yy} &= 0.
\end{aligned}$$

Here there hold $a_\sigma c_\sigma - b_\sigma^2 > 0$ for all $\sigma = 1,\ldots,n$. The solutions ζ_σ of this linear system can be considered a *saddle surfaces* $(x,y,\zeta_\sigma(x,y))$ over $B_R \subset \mathbb{R}^2$. For such surfaces we find the following result: Let $\Gamma_\sigma \subset \mathbb{R}^3$ be the spatial curves generated by $(x,y,\zeta_{R,\sigma}) \in \mathbb{R}^3$. Each of these curves satisfies a three-point-condition with a constant $\Lambda \in \mathbb{R}$ in the sense that any plane $z = \alpha x + \beta y + \gamma$ passing three different points on Γ_σ permits an uniform estimate of the form

$$\sqrt{\alpha^2+\beta^2} \le \Lambda.$$

Then by a theorem of Radó there is a constant $c_1 = c_1(\Lambda) \in (0,+\infty)$ with

$$[\zeta_\sigma]_{C^1(\mathring{B}_R)} \le c_1(\Lambda) \quad \text{for } \Lambda = 1,\ldots,n$$

for the half-norms $[\zeta_\sigma]_{C^1(\mathring{B}_R)}$, see e.g. Gilbarg and Trudinger [71]. On the other hand, Schauder [144] ensures that $\Gamma_\sigma \subset \mathbb{R}^3$ actually satisfies a three-point-condition over the strictly convex domain $B_R \subset \mathbb{R}^2$ with a constant $\Lambda = \Lambda(C_\zeta^*,R)$ which leads us to

$$[\zeta_\sigma]_{C^1(\mathring{B}_R)} \le c_2(C_\zeta^*,R) \quad \text{for all } \sigma = 1,\ldots,n.$$

This proves the statement. □

Chapter 10
Immersions with prescribed mean curvature

In this chapter we consider critical points of functionals of Gulliver type in parametric and non-parametric form. Such surfaces represent special immersions with prescribed mean curvature fields. We discuss various analytical and geometric properties, and we present a curvature estimate for graphs.

10.1 Critical points of parametric Gulliver-type functionals

Preparations

Following Gulliver [75] and Böhme, Hildebrandt and Tausch [16] we consider the following *functional of mean-curvature-type*

$$\mathscr{G}[X] := \iint_B \Big\{ \Gamma(X)W + 2X_u \circ \mathbf{A}(X) \circ X_v \Big\}\, dudv \tag{10.1}$$

which generalizes the usual area functional

$$\mathscr{A}[X] = \iint_B W\, dudv,$$

and where

$$\Gamma \in C^{3+\alpha}(\mathbb{R}^3, (0,\infty)), \quad \alpha \in (0,1)$$

is a positive weight function and

$$\mathbf{A} \in C^{3+\alpha}(\mathbb{R}^n, \mathbb{R}^{n\times n}), \quad \alpha \in (0,1)$$

a real-valued and skew-symmetric matrix.

Before we come to the computation of the Euler-Lagrange equations of critical points for $\mathscr{G}[X]$ we want to convince ourself of its geometric character.

Proposition 10.1. *The Gulliver-type functional $\mathscr{G}[X]$ is invariant w.r.t. parameter transformations of class $\mathfrak{P}$.*

Proof. It is sufficient to prove the invariance for the term $X_u \circ \mathbf{A}(X) \circ X_v$. Thus let $u^i(v^\alpha) \in \mathfrak{P}$ be such a parameter transformation.

Then we compute

$$\begin{aligned} X_{v^1} \circ \mathbf{A}(X) \circ X_{v^2} &= \left[X_{u^1} u^1_{v^1} + X_{u^2} u^2_{v^1}\right] \circ \mathbf{A}(X) \circ \left[X_{u^1} u^1_{v^2} + X_{u^2} u^2_{v^2}\right] \\ &= (u^1_{v^1} u^2_{v^2} - u^1_{v^2} u^2_{v^1}) X_{u^1} \circ \mathbf{A}(X) \circ X_{u^2}\,. \end{aligned}$$

The transformation formula for multiple integrals yields the claim. □

In particular, the form $e_1 \circ \mathbf{A}(X) \circ e_2$ does not depend on the choice of the orthonormal frame (e_1, e_2) as the following lemma emphasizes.

Lemma 10.1. *Let a point $w \in B$ be given as well as two orthonormal frames $\{e_1, e_2\}$ and $(\widetilde{e}_1, \widetilde{e}_2)$ spanning the tangential plane $\mathfrak{T}_X(w)$. Then it holds*

$$e_1 \circ \mathbf{A}(X) \circ e_2 = \widetilde{e}_1 \circ \mathbf{A}(X) \circ \widetilde{e}_2\,.$$

Proof. Consider the SO_2-action

$$\widetilde{e}_1 = \cos\varphi\, e_1 - \sin\varphi\, e_2\,, \quad \widetilde{e}_2 = \sin\varphi\, e_1 + \cos\varphi\, e_2$$

with a rotation angle φ. Then we infer

$$\widetilde{e}_1 \circ \mathbf{A}(X) \circ \widetilde{e}_2 = \cos^2\varphi\, e_1 \circ \mathbf{A}(X) \circ e_2 + \sin^2\varphi\, e_1 \circ \mathbf{A}(X) \circ e_2$$

due to the skew-symmetry of $\mathbf{A}(X)$, in particular $e_k \circ \mathbf{A}(X) \circ e_k = 0$ for $k = 1, 2$. □

Example: Hildebrandt's functional

In case $n = 1$ of one codimension, Hildebrandt in [85] considered the functional

$$\iint\limits_B \Big\{ |X_u \times X_v| + 2Q(X) \cdot (X_u \times X_v) \Big\}\, dudv$$

with a smooth vector field $Q\colon \mathbb{R}^3 \to \mathbb{R}^3$. As it will turn out later critical points $X\colon B \to \mathbb{R}^3$ have the scalar mean curvature

$$H(X) = 2\,\mathrm{div}_X Q(X) := 2\left(\frac{\partial q_1(X)}{\partial x^1} + \frac{\partial q_2(X)}{\partial x^2} + \frac{\partial q_3(X)}{\partial x^3}\right)$$

with the spatial divergence operator div_X and satisfy

$$\Delta X = 2H(X)WN \quad \text{in } B$$

using conformal parameters $(u, v) \in B$.

In terms of our skew-symmetric matrix $\mathbf{A}(X)$ we immediately verify

$$\mathbf{A}(X) = \begin{pmatrix} 0 & q_3(X) & -q_2(X) \\ -q_3(X) & 0 & q_1(X) \\ q_2(X) & -q_1(x) & 0 \end{pmatrix}.$$

Euler-Lagrange equations

We come to the calculation of the first variation of $\mathscr{G}[X]$.

Theorem 10.1. *Let a conformally parametrized immersion $X\colon B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Assume that it is critical for the functional $\mathscr{G}[X]$. Then it satisfies the Euler-Lagrange system*

$$\triangle X = 2\sum_{\sigma=1}^{n} H_\sigma W N_\sigma \quad \textit{in } B \tag{10.2}$$

with the mean curvature field

$$H_\sigma = \frac{1}{\Gamma(X)W}\sum_{k,\ell,m=1}^{n+2} a_{k\ell,x^m}(x_u^k x_v^\ell n_\sigma^m + x_u^m x_v^k n_\sigma^\ell + x_u^\ell x_v^m n_\sigma^k) + \frac{\Gamma_X(X)\cdot N_\sigma}{2\Gamma(X)} \tag{10.3}$$

for $\sigma = 1,\ldots,n$ and with the skew-symmetric matrix $\mathbf{A}(X) = (a_{k\ell}(X))_{k,\ell=1,\ldots,n}$.

Proof. In view of our results from the previous chapter it remains to evaluate the variations of $\Gamma(X)$ and the form $X_u \circ \mathbf{A}(X) \circ X_v$. For this purpose we again consider normal variations of the form

$$\widetilde{X}(u,v) = X(u,v) + \varepsilon\varphi(u,v)N_{\widehat{\gamma}}(u,v)$$

where

$$N_{\widetilde{\gamma}} = \sum_{\sigma=1}^{n} \widehat{\gamma}(u,v)N_\sigma(u,v), \quad \sum_{\sigma=1}^{n} (\widehat{\gamma}^\sigma)^2 = 1$$

with arbitrary $\varphi \in C_0^\infty(B,\mathbb{R})$ and $\varepsilon \in (-\varepsilon_0,+\varepsilon_0)$. First it holds

$$\delta\Gamma(X) := \frac{\partial}{\partial\varepsilon}\Gamma(\widetilde{X})\Big|_{\varepsilon=0} = \Gamma_X(X)\cdot N_{\widehat{\gamma}}\varphi$$

such that, together with (9.2), we arrive at

$$\delta\left[\Gamma(X)W\right] = -2\Gamma(X)H_{\widehat{\gamma}}W\varphi + \Gamma_X(X)\cdot N_{\widehat{\gamma}}W\varphi. \tag{10.4}$$

It remains to compute the first variation of $X_u \circ \mathbf{A}(X) \circ X_v$.

We calculate (note that $X_u \circ \mathbf{A}(X) \circ X_u = X_v \circ \mathbf{A}(X) \circ X_v = 0$)

$$\begin{aligned}
\delta\,[X_u \circ \mathbf{A}(X) \circ X_v] &= \delta X_u \circ \mathbf{A}(X) \circ X_v + X_u \circ \mathbf{A}(X) \circ \delta X_v + X_u \circ \delta\mathbf{A}(X) \circ X_v \\
&= \left\{\varphi_u N_{\widehat{\gamma}} + \varphi N_{\widehat{\gamma},u}\right\} \circ \mathbf{A}(X) \circ X_v + X_u \circ \mathbf{A}(X) \circ \left\{\varphi_v N_{\widehat{\gamma}} + \varphi N_{\widehat{\gamma},v}\right\} \\
&\quad + X_u \circ \delta\mathbf{A}(X) \circ X_v \\
&= \operatorname{div}\left(N_{\widehat{\gamma}} \circ \mathbf{A}(X) \circ X_v\,\varphi, X_u \circ \mathbf{A}(X) \circ N_{\widehat{\gamma}}\varphi\right) \\
&\quad - N_{\widehat{\gamma}} \circ \mathbf{A}(X)_u \circ X_v\,\varphi - X_u \circ \mathbf{A}(X)_v \circ N_{\widehat{\gamma}}\varphi + X_u \circ \delta\mathbf{A}(X) \circ X_v\,.
\end{aligned}$$

Setting $N_{\widehat{\gamma}} = (n^1_{\widehat{\gamma}}, \ldots, n^{n+2}_{\widehat{\gamma}})$ we have

$$\begin{aligned}
\sum_{k,\ell,m=1}^{n+2} X_u \circ \delta\mathbf{A}(X) \circ X_v &= a_{k\ell,x^m} n^m_{\widehat{\gamma}} x^\ell_v x^k_u\,\varphi\,, \\
\sum_{k,\ell,m=1}^{n+2} N_{\widehat{\gamma}} \circ \mathbf{A}(X)_u \circ X_v &= a_{k\ell,x^m} x^m_u x^\ell_v n^k_{\widehat{\gamma}}, \\
\sum_{k,\ell,m=1}^{n+2} X_u \circ \mathbf{A}(X)_v \circ N_{\widehat{\gamma}} &= a_{k\ell,x^m} x^m_v n^\ell_{\widehat{\gamma}} x^k_u
\end{aligned}$$

and we arrive at

$$\begin{aligned}
\delta\,[X_u \circ \mathbf{A}(X) \circ X_v] &= \operatorname{div}\left(N_{\widehat{\gamma}} \circ \mathbf{A}(X) \circ X_v\,\varphi, X_u \circ \mathbf{A}(X) \circ N_{\widehat{\gamma}}\varphi\right) \\
&\quad + \sum_{k,\ell,m=1}^{n+2} a_{k\ell,x^m}\,(x^k_u x^\ell_v n^m_{\widehat{\gamma}} - x^m_u x^\ell_v n^k_{\widehat{\gamma}} - x^k_u x^m_v n^\ell_{\widehat{\gamma}})\varphi \\
&= \operatorname{div}\left(N_{\widehat{\gamma}} \circ \mathbf{A}(X) \circ X_v\,\varphi, X_u \circ \mathbf{A}(X) \circ N_{\widehat{\gamma}}\varphi\right) \\
&\quad + \sum_{k,\ell,m=1}^{n+2} a_{k\ell,x^m}\,(x^k_u x^\ell_v n^m_{\widehat{\gamma}} + x^m_u x^k_v n^\ell_{\widehat{\gamma}} + x^\ell_u x^m_v n^k_{\widehat{\gamma}})\varphi.
\end{aligned}$$

Together with (10.4) we conclude

$$\begin{aligned}
\delta\mathscr{G}[X] &= \iint_B \operatorname{div}\left(N_{\widehat{\gamma}} \circ \mathbf{A}(X) \circ X_v\,\varphi, X_u \circ \mathbf{A}(X) \circ N_{\widehat{\gamma}}\varphi\right) dudv \\
&\quad - 2\iint_B \Gamma(X) H_{\widehat{\gamma}} W\,\varphi\,dudv + \iint_B \Gamma_X(X) \cdot N_{\widehat{\gamma}} W\,\varphi\,dudv \\
&\quad + 2\sum_{k,\ell,m=1}^{n+2} \iint_B a_{k\ell,x^m}\,(x^k_u x^\ell_v n^m_{\widehat{\gamma}} + x^m_u x^k_v n^\ell_{\widehat{\gamma}} + x^\ell_u x^m_v n^k_{\widehat{\gamma}})\varphi\,dudv
\end{aligned}$$

for all $\varphi \in C^\infty_0(B, \mathbb{R})$.

Vanishing of the first variations now means

$$H_{\widehat{\gamma}} = \frac{1}{\Gamma(X)W} \sum_{k,\ell,m=1}^{n+2} a_{k\ell,x^m}(x_u^k x_v^\ell n_{\widehat{\gamma}}^m + x_u^m x_v^k n_{\widehat{\gamma}}^\ell + x_u^\ell x_v^m n_{\widehat{\gamma}}^k) + \frac{\Gamma_X(X) \cdot N_{\widehat{\gamma}}}{2\Gamma(X)}.$$

Let $\widehat{\gamma}^\sigma \equiv 1$ for some index $\sigma \in \{1,\ldots,n\}$ and $\widehat{\gamma}^\omega \equiv 0$ for all $\omega \neq \sigma$. Then we have $N_{\widehat{\gamma}} = N_\sigma$, $n_{\widehat{\gamma}}^\ell = n_\sigma^\ell$ and $H_{\widehat{\gamma}} = H_\sigma$. We therefore conclude

$$H_\sigma = \frac{1}{\Gamma(X)W} \sum_{k,\ell,m=1}^{n+2} a_{k\ell,x^m}(x_u^k x_v^\ell n_\sigma^m + x_u^m x_v^k n_\sigma^\ell + x_u^\ell x_v^m n_\sigma^k) + \frac{\Gamma_X(X) \cdot N_\sigma}{2\Gamma(X)}.$$

This holds for all $\sigma = 1,\ldots,n$. $\square$

Invariance of the Euler-Lagrange equations

As in Lemma 10.1 we immediately verify that this special mean-curvature field $H = (H_1,\ldots,H_n)$ from the previous proof is invariant w.r.t. dilatations of the tangent vectors X_u and X_v as well as w.r.t. rotations of the basis $\{X_u, X_v\}$, i.e. there hold

$$H(X,\lambda e_1,\mu e_2,N_\sigma) = H(X,e_1,e_2,N_\sigma) \quad \text{for real } \lambda,\mu \in (0,+\infty)$$

as well as

$$H(X,\widetilde{e}_1,\widetilde{e}_2,N_\sigma) = H(X,e_1,e_2,N_\sigma).$$

Secondly, the reader is invited to verify the mean curvature H for critical points X of Hildebrandt's functional

$$\iint_B \Big\{ |X_u \times X_v| + 2Q(X) \cdot (X_u \times X_v) \Big\} \, du dv$$

from above in case $n = 1$, i.e. to prove

$$H(X) = \frac{\partial}{\partial z} q_3(X) + \frac{\partial}{\partial y} q_2(X) + \frac{\partial}{\partial x} q_1(X).$$

And finally we want to show explicitely the invariance of the mean curvature system (10.2), (10.3) w.r.t. rotations of the ONF N. For this we write (10.3) as an inner product in the form

$$\begin{aligned} H_\sigma &= \frac{1}{\Gamma(X)} \left\{ \frac{1}{W} \sum_{k,\ell,m=1}^{n+2} (a_{k\ell,x^m} + a_{\ell m,x^k} + a_{mk,\ell}) x_u^k x_v^\ell + \frac{\Gamma_X^m(X)}{2} \right\} n_\sigma^m \\ &=: \Gamma^*(X,X_u,X_v) \cdot N_\sigma \end{aligned}$$

setting $\Gamma_X = (\Gamma_X^1,\ldots,\Gamma_X^{n+2})$.

Applying then an orthogonal mapping **R** we arrive at

$$\begin{aligned}
\Delta X &= 2\sum_{\sigma=1}^{n} H_\sigma W N_\sigma = 2W\sum_{\sigma=1}^{n}\Big\{\Gamma^*(X,X_u,X_v)\cdot N_\sigma\Big\}N_\sigma \\
&= 2W\sum_{\sigma=1}^{n}\left\{\Gamma^*(X,X_u,X_v)\cdot\left(\sum_{\omega=1}^{n} r_{\sigma\omega}N_\omega\right)\right\}\sum_{\omega'=1}^{n} r_{\sigma\omega'}N_{\omega'} \\
&= 2W\sum_{\omega=1}^{n}\left(\sum_{\sigma=1}^{n} r_{\sigma\omega}^2\right)\Big\{\Gamma^*(X,X_u,X_v)\cdot N_\omega\Big\}N_\omega \\
&\quad + 2W\sum_{\substack{\omega,\omega'=1\\ \omega\neq\omega'}}^{n}\left(\sum_{\sigma=1}^{n} r_{\sigma\omega}r_{\sigma\omega'}\right)\Big\{\Gamma^*(X,X_u,X_v)\cdot N_\omega\Big\}N_{\omega'} \\
&= 2W\sum_{\omega=1}^{n}\Big\{\Gamma^*(X_u,X_u,X_v)\cdot N_\omega\Big\}N_\omega \\
&= \sum_{\omega=1}^{n} H_\omega W N_\omega\,.
\end{aligned}$$

Proposition 10.2. *The mean curvature system (10.2) together with the mean curvature field (10.3) is invariant w.r.t. the choice of an ONF N.*

Surfaces in $\mathbb{R}^3$

In particular, critical points $X\colon B\longrightarrow\mathbb{R}^3$ of the functional

$$\iint\limits_B \Big\{\Gamma(X)|X_u\times X_v| + 2Q(X)\cdot(X_u\times X_v)\Big\}\,dudv$$

have the scalar mean curvature

$$H(X) = \frac{1}{\Gamma(X)}\Big\{\Gamma_X(X)\cdot N + 2\,\mathrm{div}_X Q(X)\Big\}$$

with the unit normal vector N. By the way, the special functional

$$\iint\limits_B \Gamma(X)W\,dudv$$

possesses an interesting geometric application.

Namely it can be considered as *the area of a surface* $X\colon B \to \mathbb{R}^3$ *immersed in the threedimensional space* $\mathbb{R}^3$ *which is equipped with the non-Euclidean metric*

$$d\tilde{s}^2 = \Gamma(x^1,x^2,x^3)\Big\{(dx^1)^2+(dx^2)^2+(dx^3)^2\Big\}.$$

For example, the stereographic projection of the sphere $S^2 \subset \mathbb{R}^3$ would yield the special weight function

$$\Gamma(X) = \frac{4}{\big(1+|X|^2\big)^2}\,,$$

see e.g. Bergner [11] or Böhme, Hildebrandt and Tausch [16].

10.2 A geometric maximum principle

Following of Dierkes [42], [43] and Hildebrandt [84] we want to establish a geometric maximum principle for immersions with prescribed mean curvature fields generalizing our results for minimal surfaces from section 9.3.

Theorem 10.2. *Let the conformally parametrized immersion* $X\colon B \to \mathbb{R}^{n+2}$ *with prescribed mean curvature* $H = (H_1,\ldots,H_n)$ *w.r.t. to an ONF* N *be given. Assume*

$$h_0 \sup_{(u,v)\in B} |X(u,v)| \le \frac{1}{n}$$

is satisfied with

$$\sup_{(u,v)\in B} |H_\sigma(u,v)| \le h_0 \quad \textit{for all } \sigma = 1,\ldots,n.$$

Then it holds

$$\triangle |X|^2 \ge 0 \quad \textit{in } B.$$

In particular, the function $|X|^2$ *is subject to the maximum principle*

$$\sup_{(u,v)\in B} |X(u,v)|^2 = \sup_{(u,v)\in \partial B} |X(u,v)|^2 .$$

Proof. Namely, we compute

$$\begin{aligned}\triangle|X|^2 &= 2|\nabla X|^2 + 2X\cdot\triangle X = 2|\nabla X|^2 + 4\sum_{\sigma=1}^n H_\sigma W X\cdot N_\sigma \\ &\ge 2\left\{1-\sum_{\sigma=1}^n |H_\sigma||X|\right\}|\nabla X|^2 \ge 2\Big\{1-nh_0|X|\Big\}|\nabla X|^2,\end{aligned}$$

and the statement follows. □

Thus in case $n = 1$ of one codimension we have to assure

$$h_0 \sup_{(u,v)\in B} |X(u,v)| \le 1.$$

The limit case is represented by the half sphere with scalar mean curvature $H \equiv 1$.

10.3 A curvature estimate for surfaces with prescribed mean curvature

We want to present a curvature estimate from Bergner and Fröhlich [13] for twodimensional graphs with prescribed mean curvature fields of the special form

$$H \in C^0(\mathbb{R}^n \times S^{n+1}, \mathbb{R})$$

which additionally satisfy the following Hölder- and Lipschitz conditions:

$$\begin{aligned} &|H(X,Z)| \le h_0 \quad \text{for all } X \in \mathbb{R}^n\,,\ Z \in S^{n+1} \quad \text{as well as} \\ &|H(X_1,Z_1) - H(X_2,Z_2)| \le h_1|X_1 - X_2|^\alpha + h_2|Z_1 - Z_2| \\ &\text{for all } X_1, X_2 \in \mathbb{R}^n \text{ and } Z_1, Z_2 \in S^{n+1}\,. \end{aligned} \tag{10.5}$$

Our proof essentially uses method from Osserman [127] already applied in section 9.7 above, and from Sauvigny [140], [141]; see also Fröhlich [62].

The curvature estimate

Theorem 10.3. *Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ with prescribed mean curvature field $H = H(X,Z)$ satisfying the properties (10.5), together with an ONF N be given. Assume furthermore*

(A1) X represents a conformal reparametrization of a surface graph

$$(x,y,\zeta_1(x,y),\ldots,\zeta_n(x,y)), \quad \zeta_\sigma \in C^{3+\alpha}(\Omega,\mathbb{R}) \text{ for } \sigma = 1,\ldots,n$$

on a bounded, simply connected and regular domain $\Omega \subset \mathbb{R}^2$;

(A2) X represents a geodesic disc $\mathfrak{B}_r(X_0)$ of geodesic radius $r > 0$ with center $X_0 := X(0,\ldots,0)$, and with a real constant $d_0 > 0$, independent of the radius r, it holds the growth estimate

$$\mathscr{A}[X] \le d_0 r^2 \tag{10.6}$$

for the area of X;

(A3) at each point $w \in B$ every unit normal vector N of the surface makes an angle of at least $\omega > 0$ with the x_1-axis.

Then there exists a real constant

$$\Theta = \Theta(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha) \in (0, +\infty),$$

with the constant $h_0 \in [0, \infty)$ from the previous section, such that it holds the curvature estimate

$$\kappa_{\sigma,1}(0,0)^2 + \kappa_{\sigma,2}(0,0)^2 \le \frac{1}{r^2}\Big\{(h_0 r)^2 + \Theta\Big\} \tag{10.7}$$

for the principal curvature $\kappa_{\sigma,1}$ und $\kappa_{\sigma,2}$ w.r.t. the ONF N and for all $\sigma = 1, \ldots, n$.

Remarks

Before we come to the proof of our curvature estimate we briefly want to discuss the assumptions of the theorem.

1. The regularity conditions (10.5) can particularly be verified by means of the mean curvature field
$$H(X,Z) = \frac{\Gamma_X(X) \cdot Z}{2\Gamma(X)}$$
for critical points of the Gulliver-type functional
$$\mathscr{G}[X] = \iint\limits_B \Gamma(X) W \, dudv,$$
at least if $\Gamma_X(X)$ and $\Gamma(X)$ are regular enough.
2. Growth estimates for the area as assumed in (10.6) can be found in chapter 14. For the purpose of a curvature estimate the quality of the constant d_0 is inessential; in particular it could also depend on the radius r. But later when we apply our curvature estimate to establish a theorem of Bernstein type we are forced to require the growth estimate (10.6) in its sharp form.
3. The condition (A3) comes from Osserman's curvature estimate presented in section 9.7. Let us verify it by means of the minimal graph (w, w^2) for $w \in \overline{B}_R$. For this purpose we introduce the two unit normal vectors
$$N_1 := \frac{1}{\sqrt{1+4u^2+4v^2}}\,(-2u, 2v, 1, 0),$$
$$N_2 := \frac{1}{\sqrt{1+4u^2+4v^2}}\,(-2v, -2u, 0, 1).$$

Along the line $\{v=0\}$ we especially infer

$$|N_1\cdot(1,0,\ldots,0)|=\frac{2|u|}{\sqrt{1+4u^2}}\longrightarrow 1\quad\text{for } |u|\to\infty.$$

Thus (A3) *is not fulfilled* for $R\to\infty$ since according to this condition it should hold contradictorily

$$|N_1\cdot(1,0,\ldots,0)|\le\cos\omega<1.$$

Unfortunately we can not apply our theorem.

Proof of the curvature estimate

First it holds

$$\kappa_{\sigma,1}(0,0)^2+\kappa_{\sigma,2}(0,0)^2 = 4H_\sigma(0,0)^2-2K_\sigma(0,0) \le 4h_0^2+2|K_\sigma(0,0)|$$

$$=\frac{1}{r^2}\Big\{(2h_0r)^2+r^2|K_\sigma(0,0)|\Big\}$$

for $\sigma=1,\ldots,n$. Then the desired estimate follows from the identity

$$K_\sigma(0,0)=\frac{(X_{uu}\cdot N_\sigma)(X_{vv}\cdot N_\sigma)-(X_{uv}\cdot N_\sigma)^2}{W^2}\Big|_{(u,v)=(0,0)}$$

if an upper and a lower bounds for the nominator resp. the denominator of the fraction on the right hand side are proved.

1. In the first part of our proof we show

$$\frac{W(w)}{r^2}\ge C_1\quad\text{for } w\in B_{\frac{1}{2}}(0,0)$$

with a constant $C_1=C_1(h_0r,d_0,\sin\omega,n)>0$.

1.1 As usual we construct a global orthonormal normal frame N as follows: The basis vectors

$$e_3:=(0,0,1,0,\ldots,0),\ldots,e_{n+2}:=(0,\ldots,0,1)$$

are not tangential to the surface. Its projections

$$N_1^*:=e_3-\frac{e_3\cdot X_u}{|X_u|^2}X_u-\frac{e_3\cdot X_v}{|X_v|^2}X_v\,,\quad N_2^*:=e_4-\frac{e_4\cdot X_u}{|X_u|^2}X_u-\frac{e_4\cdot X_v}{|X_v|^2}X_v$$

etc. can be transformed into such an ONF $\{N_1,\ldots,N_n\}$. We use this frame in the first part of the proof.

1.2 Using conformal parameters $(u,v) \in B$ we recall the estimate

$$|\triangle X(u,v)| \le nh_0 |\nabla X(u,v)|^2 \quad \text{in } B. \tag{10.8}$$

1.3 Assumption (A2) yields: Let $\Gamma(B)$ denotes the set of all continuous and piecewise continuously differentiable curves $\gamma\colon[0,1] \to \overline{B}$ with $\gamma(0)=(0,0)$ and $\gamma(1) \in \partial B$. Then it holds

$$\inf_{\gamma\in\Gamma(B)} \int_0^1 \left| \frac{d}{dt} X \circ \gamma(t) \right| dt \ge r.$$

1.4 Assumption (A3) ensures the estimate

$$|\nabla x^1|^2 \ge W \sin^2\omega \quad \text{in } B,$$

see the proof of Osserman's curvature estimate from section 9.7. This inequality actually is true for immersions which are not necessarily minimal for its proof makes only use of the conformality relations.

To construct a lower bound for the area element we will employ various results from Heinz [80], see section 9.4.

1.5 We define the plane mapping

$$F^*(u,v) := (x^1(u,v), x^2(u,v))\colon B \longrightarrow \mathbb{R}^2 .$$

(i) First we infer

$$|\Delta F^*(w)| \le \frac{2nh_0}{\sin^2\omega}\, |\nabla F^*(w)|^2 \quad \text{for all } w \in B$$

taking

$$|\Delta F^*| \le |\Delta X| \le nh_0 |\nabla X|^2 = 2nh_0 W \le \frac{2nh_0}{\sin^2\omega}\, |\nabla x^1|^2$$
$$\le \frac{2nh_0}{\sin^2\omega}\, |\nabla F^*|^2$$

into account;

(ii) Secondly it holds

$$|\nabla X(w)|^2 \le \frac{2}{\sin^2\omega}\, |\nabla F^*(w)|^2 \quad \text{for all } w \in B$$

what follows from

$$|\nabla X|^2 = 2W \le \frac{2}{\sin^2\omega}\, |\nabla x^1|^2 \le \frac{2}{\sin^2\omega}\, |\nabla F^*|^2 .$$

1.6 Now let $w_0 \in B$ and $\nu \in (0,1)$ such that

$$\mathring{B}_{2\nu}(w_0) := \left\{w \in B : |w - w_0| < 2\nu\right\} \subset \mathring{B}.$$

We consider

$$Y(w) := \frac{1}{r}\{X(w_0 + 2\nu w) - X(w_0)\}, \quad w \in B,$$

as well as

$$F(w) := (y^1(w), y^2(w)) \colon B \longrightarrow \mathbb{R}^2.$$

Again there hold

$$|Y_u(w)|^2 = \frac{4\nu^2}{r^2} W(w_0 + 2\nu w) = |Y_v(w)|^2, \quad Y_u(w) \cdot Y_v(w) = 0,$$

and due to (10.8) we infer

$$|\triangle Y(w)| \le n(h_0 r)|\nabla Y(w)|^2 \quad \text{in } \mathring{B}.$$

1.7 Together with point 1.5(ii) of the proof we conclude

$$\begin{aligned} |\triangle F(w)| &\le |\triangle Y(w)| \le n(h_0 r)|\nabla Y(w)|^2 = \frac{8n\nu^2(h_0 r)}{r^2} W(w_0 + 2\nu w) \\ &= \frac{4n\nu^2(h_0 r)}{r^2} |\nabla X(w_0 + 2\nu w)|^2 \le \frac{8n\nu^2(h_0 r)}{r^2 \sin^2 \omega} |\nabla F^*(w_0 + 2\nu w)|^2 \\ &= \frac{8n\nu^2(h_0 r)}{r^2 \sin^2 \omega} \frac{r^2}{4\nu^2} |\nabla F(w)|^2 \le \frac{2n(h_0 r)}{\sin^2 \omega} |\nabla F(w)|^2 \end{aligned} \tag{10.9}$$

for all $w \in \mathring{B}$. Due to (A1) the mapping $F = F(u,v)$ is one-to-one with positive Jacobian $J_F(w) > 0$ in $\mathring{B}$. Furthermore, (A2) gives

$$\mathscr{D}[F] \le \mathscr{D}[Y] \le \frac{1}{r^2} \mathscr{D}[X] \le 2d_0$$

for the Dirichlet energy of F given by

$$\mathscr{D}[F] = \iint\limits_B \left\{|F_u|^2 + |F_v|^2\right\} du dv.$$

Following now Heinz [80], Theorem 6 (see section 9.4 above) there is a constant $c_1 = c_1(h_0 r, d_0, \sin\omega, n) \in (0, +\infty)$ such that

$$|\nabla F(u,v)| \le c_1(h_0 r, d_0, \sin\omega, n) \quad \text{for all } (u,v) \in \mathring{B}_{\frac{1}{2}}(0,0).$$

1.8 From (10.9) we infer

$$\frac{1}{r^2}W(w_0+2\nu w) \le \frac{1}{4\nu^2\sin^2\omega}\,c_1(h_0r,d_0,\sin\omega,n)^2 =: c_2(h_0r,d_0,\sin\omega,\nu,n)$$

for all $w \in \mathring{B}_{\frac{1}{2}}(0,0)$. In particular, it holds

$$\frac{1}{r^2}W(w) \le c_2(h_0r,d_0,\sin\omega,\nu,n) \quad \text{for all } w \in B_\nu(w_0) \tag{10.10}$$

and all $w_0 \in \mathring{B}$ mit $\mathring{B}_{2\nu}(w_0) \subset \mathring{B}$. Later we will make use of this estimate.

1.9 Together with

$$J_F(w) > 0 \quad \text{and} \quad \mathscr{D}[F] \le 2d_0$$

we apply Heinz [80], Lemma 17[1]: There exists a constant $c_3 \in (0,\infty)$ such that

$$|\nabla F(w)|^2 \le c_3(h_0r,d_0,\sin\omega,n)|\nabla F(0,0)|^{\frac{2}{5}} \quad \text{for all } w \in \mathring{B}_{\frac{1}{2}}(0,0).$$

Together with (10.9) we thus have

$$\begin{aligned}
\frac{4\nu^2}{r^2}W(w_0+2\nu w) &\le \frac{1}{\sin^2\omega}|\nabla F(w)|^2 \le \frac{c_3(h_0r,d_0,\sin\omega,n)}{\sin^2\omega}|\nabla F(0,0)|^{\frac{2}{5}} \\
&\le \frac{c_3(h_0r,d_0,\sin\omega,n)}{\sin^2\omega}|\nabla Y(0,0)|^{\frac{2}{5}} \\
&= \frac{c_3(h_0r,d_0,\sin\omega,n)}{\sin^2\omega}\left[\frac{8\nu^2}{r^2}W(w_0)\right]^{\frac{1}{5}},
\end{aligned}$$

and rearranging proves the following inequality of Harnack type

$$\left[\frac{W(w_0)}{r^2}\right]^{\frac{1}{5}} \ge \frac{4\cdot 8^{-\frac{1}{5}}\nu^{\frac{8}{5}}\sin^2\omega}{c_3(h_0r,d_0,\sin\omega,n)}\,\frac{W(w_0+2\nu w)}{r^2} \quad \text{for all } w \in \mathring{B}_{\frac{1}{2}}(0,0).$$

---

[1] Let the mapping $z(w) = (x(w),y(w)) \in C^2(\mathring{B},\mathbb{R})$ satisfy the differential inequality

$$|\Delta z| \le a(|z_u|^2+|z_v|^2) + b(|z_u|+|z_v|)$$

with positive constants a and b. Furthermore, let $x_uy_v - x_vy_u \ne 0$ and $\mathscr{D}[z] \le N < \infty$. Then there hold

$$c_1(a,b,N,r)\big\{|z_u(0,0)|+|z_v(0,0)|\big\}^{\frac{1+3r}{1-r}} \le |z_u(w)|+|z_v(w)| \le c_2(a,b,N,r)\big\{|z_u(0)|+|z_v(0)|\big\}^{\frac{1-r}{1+3r}}$$

for all $|w| \le r < 1$.

Equivalently we arrive at

$$c_4(h_0 r, d_0, \sin\omega, \nu, n)\left[\frac{W(w)}{r^2}\right]^5 \leq \frac{W(w_0)}{r^2} \quad \text{for all } w \in \mathring{B}_\nu(w_0)$$

with the real constant

$$c_4(h_0 r, d_0, \sin\omega, \nu, n) := \frac{2^7 \nu^8 \sin^{10}\omega}{c_3(h_0 r, d_0, \sin\omega, n)^5} \in (0, +\infty).$$

1.10 Now we prove the existence of a point $w^* \in B_{1-\nu_0}(0,0)$, where we introduce $\nu_0 := \min(e^{-4\pi d_0}, \frac{1}{2})$, such that

$$\frac{W(w^*)}{r^2} \geq \frac{1}{4(1-e^{-4\pi d_0})} =: c_5(d_0) > 0.$$

For this purpose we apply the lemma Courant and Lebesgue considering assumption (A2): For given $\delta := e^{-4\pi d_0}$ we find a $\delta^* \in [\delta, \sqrt{\delta}]$ such that

$$\int\limits_{|w-w_0|=\delta^*} |dX(w)| \leq 2\sqrt{\frac{\pi d_0 r^2}{-\log\delta}} = 2\sqrt{\frac{\pi d_0 r^2}{4\pi d_0}} = r.$$

Recall point 1.3 of the proof and consider a curve $\gamma_1(t)$ in B from the origin $(0,0)$ to $w_1 := (1-\delta^*)w \in B$ satisfying

$$\gamma_1(t) := w_0 t, \quad t \in [0, 1-\delta^*].$$

At the same time consider curves

$$\gamma_2(t) := w_0 + (w - 1 - w_0)e^{\pm it}, \quad t \in [0, t_1(\delta^*)],$$

starting in w_1, where $t_1(\delta^*)$ is chosen such that $\gamma_2(t)$ ends on ∂B, and this curve runs along the arc $|w - w_0| = \delta^*$, $w \in B$. Furthermore, specify $\gamma_2(t)$ to ensure

$$\int\limits_0^{t_1(\delta^*)} \left|\frac{d}{dt} X(\gamma_2(t))\right| dt \leq \frac{r}{2}.$$

Combine $\gamma_1(t)$ and $\gamma_2(t)$ to a continuous and piecewise continuously differentiable curve $\gamma(t)$ and compute with a suitable $t_0 \in [0, 1-\delta^*]$

$$\begin{aligned} r &\leq \int\limits_0^{1-\delta^*} \left|\frac{d}{dt} X(\gamma_1(t))\right| dt + \int\limits_0^{t_1(\delta^*)} \left|\frac{d}{dt} X(\gamma_2(t))\right| dt \\ &\leq (1-\delta)\left|\frac{d}{dt} X(\gamma_1(t_0))\right| + \frac{r}{2}. \end{aligned}$$

Rearranging yields

$$\left|\frac{d}{dt}X(\gamma_1(t_0))\right| \geq \frac{r}{2(1-\delta)} = \frac{r}{2(1-e^{-4\pi d_0})}$$

proving the existence of the desired point $w^* \in B_{1-v_0}(0,0)$.

1.11 Using the above Harnack inequality (with the setting $v := \frac{1}{2}v_0 < \frac{1}{4}$) we arrive iteratively at

$$\frac{W(w_0)}{r^2} \geq c_4^{1+5+5^2+\ldots+5^{m-1}} c_5(d_0)^{5^m} =: C_1(h_0 r, d_0, \sin\omega, n) > 0$$

for all $w_0 \in B_{1-v_0}(0,0)$. We infer

$$\frac{W(w)}{r^2} \geq C_1(h_0 r, d_0, \sin\omega, n) \quad \text{for all } w \in B_{\frac{1}{2}}(0,0)\,. \tag{10.11}$$

This determines the first part of our proof.

2. The second part of the proof is devoted to establishing an upper bound for the second derivatives of $X = X(u,v)$ making use of the elliptic system

$$\triangle X = 2H(X,N_1)WN_1 + 2H(X,N_2)WN_2 + \ldots + 2H(X,N_n)WN_n\,.$$

For this purpose we construct a suitable ONF $N = (N_1,\ldots,N_n)$.

2.1 First we define

$$Z(u,v) = \frac{1}{r}\{X(u,v) - X(0,0)\} = \frac{1}{r}X(u,v), \quad (u,v) \in B.$$

Let W_Z denote the area element of the mapping $Z = Z(u,v)$. Then we have

$$|Z_u|^2 = W_Z = |Z_v|^2 \quad \text{und} \quad Z_u \cdot Z_v = 0 \quad \text{in } B.$$

Obviously it holds $r^2 W_Z = W_X$ with $W_X := |X_u|^2 = |X_v|^2$, and we compute

$$\begin{aligned}\triangle Z &= \frac{2}{r}H(X,N_1)W_X N_1 + \ldots + \frac{2}{r}H(X,N_n)W_X N_n \\ &= 2rH(rZ,N_1)W_Z N_1 + \ldots + 2rH(rZ,N_n)W_Z N_n\,.\end{aligned}$$

2.2 From the estimate (10.10) we infer

$$|\triangle Z(w)| \leq 2(n-2)(rh_0)c_2(h_0 r, d_0, \sin\omega, n) \tag{10.12}$$

for all $w \in B_{\frac{1}{2}}(0,0)$.

Due to $Z(0,0) = (0,0,0)$ we have furthermore

$$|Z(u,v)| = |Z(u,v) - Z(0,0)| \le 2 \max_{w \in B_{\frac{1}{2}}(0,0)} |\nabla Z(w)|$$
$$\le 2\sqrt{2c_2(h_0 r, d_0, \sin\omega, n)} \qquad \text{in } B_{\frac{1}{2}}(0,0).$$

Potential theoretic estimates (see e.g. Sauvigny [143], Kapitel XII, §2, Satz 2[2] yield a real constant $c_6(h_0 r, d_0, \sin\omega, n, \alpha)$ with the property

$$|Z_{u^i}(w_1) - Z_{u^i}(w_2)| \le c_6(h_0 r, d_0, \sin\omega, n, \alpha)|w_1 - w_2|^\alpha$$
$$\text{for } w_1, w_2 \in B_{\frac{1}{4}}(0,0) \tag{10.13}$$

and for all $\alpha \in (0,1)$. Thus we arrive at

$$|W_Z(w_1) - W_Z(w_2)| \le c_7(h_0 r, d_0, \sin\omega, n, \alpha)|w_1 - w_2|^\alpha$$
$$\text{for all } w_1, w_2 \in B_{\frac{1}{4}}(0,0)$$

with a constant $c_7 := 4\sqrt{c_2}\, c_6$.

2.3 Using the mean value theorem we arrive at the following Lipschitz estimate

$$|Z(w_1) - Z(w_2)| \le c_8(h_0 r, d_0, \sin\omega, n, \alpha)\, |w_1 - w_2| \quad \text{for } w_1, w_2 \in B_{\frac{1}{2}}(0,0).$$

In a neighborhood of the origin we now construct an ONF N with controlled Hölder norm.

2.4 For this purpose we choose unit normal vectors $\overline{N}_1, \ldots, \overline{N}_n \in \mathbb{R}^{n+2}$ such that

$$\overline{N}_\sigma \cdot Z_{u^j}(0,0) = 0, \quad \overline{N}_\sigma \cdot \overline{N}_\omega = \delta_{\sigma\omega}, \quad j = 1,2, \quad \sigma, \omega = 1, \ldots, n. \tag{10.14}$$

We define

$$N_\sigma^*(w) := \overline{N}_\sigma - \frac{\overline{N}_\sigma \cdot Z_u(w)}{|Z_u(w)|^2}\, Z_u(w) - \frac{\overline{N}_\sigma \cdot Z_v(w)}{|Z_v(w)|^2}\, Z_v(w) \quad \text{in } \mathring{B}.$$

2.5 These vectors belong to normal space of $Z(w)$ but eventually they are not linearly independent.

[2] Let $X \in C^2(B, \mathbb{R}^n)$ solve $|\Delta X| \le a|\nabla X|^2 + b$ in B such that $aM < 1$ with $M := \sup_{w \in B} |X(w)|$. Let $\alpha \in (0,1)$. Then it holds
$$\|X\|_{C^{1+\alpha}(B_{1-\varepsilon}(0,0))} \le C(a,b,M,\varepsilon,\alpha).$$

We determine a $\nu_1 = \nu_1(h_0 r, d_0, \sin\omega, n, \alpha) > 0$ with the property

$$|N_\sigma^*(w)|^2 = 1 - \frac{[\overline{N}_\sigma \cdot Z_u(w)]^2}{W_Z(w)} - \frac{[\overline{N}_\sigma \cdot Z_v(w)]^2}{W_Z(w)} \geq \frac{1}{2} \quad \text{in } \mathring{B}_{\nu_1}(0,0). \quad (10.15)$$

Together with (10.14) and (10.13) we calculate

$$\begin{aligned} |\overline{N}_\sigma \cdot Z_{u^\ell}(w)|^2 &= |\overline{N}_\sigma \cdot \{Z_{u^\ell}(w) - Z_{u^\ell}(0,0)\}|^2 \\ &\leq |Z_{u^\ell}(w) - Z_{u^\ell}(0,0)|^2 \\ &\leq c_6(h_0 r, d_0, \sin\omega, n, \alpha)^2 |w|^{2\alpha} \end{aligned}$$

for $\ell = 1, 2$, $\sigma = 1, \ldots, n$, and from (10.11) we deduce

$$W_Z(w) \geq C_1(h_0 r, d_0, \sin\omega, n) \quad \text{in } \mathring{B}_{\frac{1}{2}}(0,0).$$

Thus it holds (10.15) if $\nu_1^{2\alpha} \leq \frac{C_1}{4c_6}$.

2.6 Note that for the vectors $N_\sigma^*(w)$, $\sigma = 1, \ldots, n$, the Hölder estimate

$$|N_\sigma^*(w_1) - N_\sigma^*(w_2)| \leq c_9(h_0 r, d_0, \sin\omega, n, \alpha)|w_1 - w_2|^\alpha$$

for $w_1, w_2 \in \mathring{B}_{\nu_1}(0,0)$ with a constant $c_9(h_0 r, d_0, \sin\omega, n, \alpha)$ are true. We infer this estimate from the Hölder estimates for Z_{u^j} and the lower bound for W_Z.

2.7 For $\sigma = 1, \ldots, n$ we define

$$\widetilde{N}_\sigma(w) := \frac{N_\sigma^*(w)}{|N_\sigma^*(w)|} \quad \text{in } \mathring{B}_{\nu_1}(0,0).$$

These vectors are well-defined since

$$|N_\sigma^*(w)|^2 \geq \frac{1}{2} \quad \text{in } B_{\nu_1}(0,0)$$

but they are not necessarily orthogonal. But note that

$$N_\sigma^* \cdot N_\omega^* = \frac{(\overline{N}_\sigma \cdot Z_u)(\overline{N}_\omega \cdot Z_u)}{W_Z} + \frac{(\overline{N}_\sigma \cdot Z_v)(\overline{N}_\omega \cdot Z_v)}{W_Z} \quad \text{for } \sigma \neq \omega$$

from where we conclude

$$\begin{aligned} |\widetilde{N}_\sigma \cdot \widetilde{N}_\omega| = \frac{|N_\sigma^* \cdot N_\omega^*|}{|N_\sigma^*||N_\omega^*|} &\leq \frac{2}{C_1}\Big\{|\overline{N}_\sigma \cdot Z_u||\overline{N}_\omega \cdot Z_u| + |\overline{N}_\sigma \cdot Z_v||\overline{N}_\omega \cdot Z_v|\Big\} \\ &\leq \frac{4c_6^2}{C_1}|w|^{2\alpha}. \end{aligned}$$

2.8 Thus we find a $\nu_2 = \nu_2(h_0 r, d_0, \sin\omega, n, \alpha)$ with $0 < \nu_2 \le \nu_1$ such that the following vectors are well-defined in $\mathring{B}_{\nu_2}(0,0)$:

$$
\begin{aligned}
N_1 &:= \widetilde{N}_1, \\
N_2 &:= \frac{\widetilde{N}_2 - \{N_1 \cdot \widetilde{N}_2\} N_1}{\sqrt{1 - \{N_1 \cdot \widetilde{N}_2\}^2}}, \\
&\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
N_n &:= \frac{\widetilde{N}_n - \{N_1 \cdot \widetilde{N}_n\} N_1 - \ldots - \{N_{n-1} \cdot \widetilde{N}_n\} N_{n-1}}{\sqrt{1 - \{N_1 \cdot \widetilde{N}_n\}^2 - \ldots - \{N_{n-1} \cdot \widetilde{N}_n\}^2}}.
\end{aligned}
\tag{10.16}
$$

Namely, choose $\nu_2 \in (0,1)$ sufficiently small to ensure that each denominator in (10.16) is greater or equal to $\frac{1}{2}$. These vectors form an ONF $\mathring{B}_{\nu_2}(0,0)$. Additionally there hold the Hölder estimates

$$
\begin{aligned}
&|N_\sigma(w_1) - N_\sigma(w_2)| \le c_{10}(h_0 r, d_0, \sin\omega, n, \alpha)|w_1 - w_2|^\alpha \\
&\text{for } w_1, w_2 \in B_{\nu_2}(0,0)
\end{aligned}
$$

for $\sigma = 1, \ldots, n$ with a constant $c_{10}(h_0 r, d_0, \sin\omega, n, \alpha)$ which can be deduced from the Hölder estimates of the N_σ^*.

2.9 Now we make use of the differential system

$$\triangle Z = 2rH(rZ, N_1) W_Z N_1 + \ldots + 2rH(rZ, N_n) W_Z N_n \quad \text{in } B_{\nu_2}(0,0).$$

We already know

$$|\triangle Z(w)| \le 2n(h_0 r) c_2 \quad \text{in } B_{\nu_2}(0,0),$$

see (10.12). Together with (10.5) we get

$$
\begin{aligned}
&|H(rZ(w_1), N_\sigma(w_1)) - H(rZ(w_2), N_\sigma(w_2))| \\
&\quad \le h_1 r^\alpha |Z(w_1) - Z(w_2)|^\alpha + h_2 |N_\sigma(w_1) - N_\sigma(w_2)| \\
&\quad \le h_1 4^\alpha r^\alpha c_8^\alpha |w_1 - w_2|^\alpha + h_2 c_{10} |w_1 - w_2|^\alpha.
\end{aligned}
$$

Therefore we find a real constant $c_{11} = c_{11}(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha)$ such that it holds

$$|\triangle Z(w_1) - \triangle Z(w_2)| \le c_{10} |w_1 - w_2|^\alpha \quad \text{for } w_1, w_2 \in B_{\nu_2}(0,0).$$

2.10 Let $v_3 := \frac{1}{2}v_2$. From the inner Schauder estimates (see e.g. Gilbarg and Trudinger [71]) we infer a real constant $C_2 \in (0,+\infty)$ with the property

$$|Z_{uu}(w)|, |Z_{uv}(w)|, |Z_{vv}(w)| \leq C_2(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha)$$

in $B_{v_3}(0,0)$.

2.11 Now from the beginning of our proof we recall

$$\begin{aligned} &\kappa_{\Sigma,1}(0,0)^2 + \kappa_{\sigma,2}(0,0)^2 \\ &\qquad \leq \frac{1}{r^2}\left\{(h_0 r)^2 + \frac{|Z_{uu}(0,0)||Z_{vv}(0,0)| + |Z_{uv}(0,0)|^2}{W_Z(0,0)^2}\right\}. \end{aligned}$$

If we set

$$\Theta(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha) := \frac{2C_2^2}{C_1^2}$$

we find

$$\kappa_{\sigma,1}(0,0)^2 + \kappa_{\sigma,2}(0,0)^2 \leq \frac{1}{r^2}\Big\{(h_0 r)^2 + \Theta\Big\}.$$

The proof is complete. □

10.4 A theorem of Bernstein type

The Bernstein type result

This curvature estimate enables us to prove the following theorem of Bernstein type.

Consider a minimal graph

$$(x, y, \varphi_1, (x,y), \ldots, \varphi_n(x,y)), \quad (x,y) \in \mathbb{R}^2.$$

Because its Gaussian curvature is non-positive, by Hadamard's theorem we can introduce geodesic discs $\mathfrak{B}_r(X_0)$ for all $X_0 = (x_0, y_0, \varphi_1(x_0,y_0), \ldots, \varphi_n(x_0,y_0))$ and all $r > 0$. Then the limit $r \to \infty$ yields the

Corollary 10.1. *Let $X = X(x,y)$, $(x,y) \in \mathbb{R}^2$, be a complete minimal graph with the properties*

(i) there exists $X_0 = (x_0, y_0, \varphi_1(x_0,y_0), \ldots, \varphi_n(x_0,y_0))$ and a radius $r_0 > 0$ such that all geodesic discs $\mathfrak{B}_r(X_0)$ with center X_0 and radius $r \geq r_0$ satisfy

$$\text{Area}[\mathfrak{B}_r(X_0)] \leq d_0 r^2 \quad \textit{for all } r \geq r_0 \tag{10.17}$$

for their area Area $[\mathfrak{B}_r(X_0)]$ with a constant $d_0 \in (0,+\infty)$ which does not depend on the radius r;

(ii) each normal vector of the graph makes an angle of at least $\omega > 0$ with the x_1-axis.

Then $X = X(x,y)$ is a linear mapping.

Proof. For any point $X_1 = (x_1, y_1, \varphi_1(x_1,y_1), \dots, \varphi_n(x_1,y_1))$ on the graph we have

$$\text{Area}[\mathfrak{B}_r(X_1)] \le 4d_0 r^2 \quad \text{for all } r \ge \max\{r_0, d(X_0,X_1)\} \tag{10.18}$$

where $d(X_0,X_1) \ge 0$ is the inner distance between X_0 and X_1 on the surface. This holds because of the inclusion

$$\mathfrak{B}_r(X_1) \subset \mathfrak{B}_{2r}(X_0) \quad \text{for all } r \ge \max\{r_0, d(X_0,X_1)\} \tag{10.19}$$

and assumption (i). Since $K \le 0$ for the Gaussian curvature we can consider geodesic discs $\mathfrak{B}_r(X_1)$ for all $r \in (0,+\infty)$ on account of Hadamard's theorem. Introduce conformal parameters into such a geodesic disc. Using our curvature estimate and letting $r \to \infty$ shows that all principal curvatures at X_1 vanish which proves the Corollary (note that Θ does not depend on r since $h_0, h_1, h_2 = 0$). □

Remarks

We want to conclude this chapter with the following remarks.

1. Osserman's curvature estimate from the previous chapter *does not need a growth estimate on the surface area.*
2. Jost and Xin [104] proved a curvature estimates for submanifolds with parallel mean curvature fields. Due to the higher dimension of the manifolds itself the authors assume a-priori bounds for the gradients. Consult also the references therein.
3. Curvature estimates and related Bernstein type result for minimal submanifolds can also be found in Smoczyk, Wang and Xin [149] where the authors extend methods from Schoen, Simon and Yau [145] for minimal immersions with vanishing normal sectional curvature; see also Ecker [50] or Wang [162].
4. Our method of proof uses essentially results from Heinz [80], and follows Sauvigny [141] where curvature estimates for two-dimensional immersions of mean curvature type in $\mathbb{R}^3$ where established.

Chapter 11

Crystalline functionals in $\mathbb{R}^3$

In this chapter we consider general elliptic functionals and its critical points in Euclidean space $\mathbb{R}^3$. The Lagrangian densities of these functionals depend on the surface vector X as well as the normal direction $X_u \times X_v$.

11.1 Examples of parametric variational problems

In the following we study critical points $X\colon B \to \mathbb{R}^3$ of parametric variational problems of the form

$$\mathscr{B}[X] := \iint_B F(X, X_u \times X_v)\, du dv \longrightarrow \text{extr!}$$

The case of general case of codimensions $n \geq 1$ is the subject in chapter 12.

But before we go further into details we want to consider some special parametric functionals to get a first impression of its broad range of applications.

Minimal surfaces

An immersion $X\colon B \to \mathbb{R}^3$ is a minimal surface if its scalar mean curvature H vanishes identically. As we have seen minimal surfaces are critical points of the area functional

$$\mathscr{A}[X] = \iint_B |X_u \times X_v|\, du dv.$$

With the setting

$$F(X, Z) = |Z|$$

the area functional turns out to be a special parametric functional of the form $\mathscr{B}[X]$.

A famous minimal surface already found by the Swiss mathematician Leonhard Paul Euler (*1707 in Basel; †1783 in St. Petersburg) is the *catenoid*

$$X(u,v) = (\cosh u \cos v, \cosh u \sin v, u) \in \mathbb{R}^3$$

which is generated by rotating the hyperbolic cosine, the so-called *catenary curve*, about the x-axis.

Note that a full rotation would generate a surface of higher topological type, thus we rather consider X restricted to a suitable simply connected domain in $\mathbb{R}^2$.

The catenoid representation $X(u,v)$ is conformal, i.e. there hold

$$X_u^2 = \sinh^2 u, \quad X_v^2 = 1+\cosh^2 u = \sinh^2 u, \quad X_u \cdot X_v = 0.$$

Furthermore, it can be deformed continuously into the *minimal helicoid*

$$X^*(u,v) = (\sinh u \sin v, -\sinh u \cos v, v)$$

as follows

$$\begin{aligned} x_\alpha(u,v) &:= \cos\alpha \sinh u \sin v + \sin\alpha \cosh u \cos v, \\ y_\alpha(u,v) &:= -\cos\alpha \sinh u \cos v + \sin\alpha \cosh u \sin v, \\ z_\alpha(u,v) &:= v\cos\alpha + u \sin\alpha. \end{aligned}$$

The helicoid is given for $\alpha = 0$, and $\alpha = \frac{\pi}{2}$ represents the catenoid. The surface parametrizations

$$X_\alpha(u,v) = \big(x_\alpha(u,v), y_\alpha(u,v), z_\alpha(u,v)\big)$$

are conformal for all α. This deformation is a special case of a general result by Bour (see Strubecker [152].

The helicoid was discovered by the French mathematician Jean Baptiste Maria Charles Meusnier de la Place (*1754 in Tours; †1793 in le Pont de Cassel).

Other important examples of minimal surfaces are the *Scherk surface*

$$X(u,v) = (u, v, \log\cos u + \log\cos v)$$

found by H. Scherk in 1834, the *Henneberg surface*

$$\begin{aligned} x(u,v) &= 2\sinh u \cos v - \frac{2}{3}\sinh(3u)\cos(3v), \\ y(u,v) &= 2\sinh u \sin v - \frac{2}{3}\sinh(3u)\sin(3v), \\ z(u,v) &= 2\cosh(2u)\cos(2v) \end{aligned}$$

named after E.L. Henneberg who found it in 1875, *E.C. Catalan's surface*

$$x(u,v) = u - \sin u \cosh v, \quad y(u,v) = 1 - \cos u \cosh v, \quad z(u,v) = 4\sin\frac{u}{2}\sinh\frac{v}{2}$$

and *A. Enneper's surface*

$$x(u,v) = u - \frac{1}{3}u^3 + uv^2, \quad y(u,v) = -v - u^2 v + \frac{1}{3}v^3, \quad z(u,v) = u^2 - v^2.$$

By the way, the later one is an *algebraic surface* since it can be rewritten in the following algebraic form

$$\left[y^2 - x^2 + \frac{4}{3}z + \frac{4}{9}z^3\right]^3 = 3z\left[y^2 - x^2 + \frac{8}{9}z - z\left(x^2 + y^2 + \frac{8}{9}z^2\right)\right],$$

see Nitsche [126].

An abundance of examples and simulations of minimal surfaces can be found in various textbooks and in the world wide net.

Immersions with prescribed mean curvature

We also considered parametric functionals whose *critical points possess non-vanishing mean curvature fields*, for example

$$\iint_B \left\{|X_u \times X_v| + \frac{2h_0}{3} X \cdot (X_u \times X_v)\right\} du dv$$

with the volume constraint

$$\frac{2h_0}{3} \iint_B X \cdot (X_u \times X_v)\, du dv = 1.$$

Critical points have constant scalar mean curvature $H \equiv h_0$. We have

$$F(X,Z) = |Z| + \frac{2h_0}{3} X \cdot Z.$$

Or, more generally, *Hildebrandt's functional*

$$\iint_B \left\{|X_u \times X_v| + 2Q(X) \cdot (X_u \times X_v)\right\} du dv$$

with the constraint

$$2 \iint_B Q(X) \cdot (X_u \times X_v)\, du dv = 1$$

for some given vector field $Q(X) = (q_1(X), q_2(X), q_3(X))$. Critical points are immersions with mean curvature

$$H(X) = \frac{\partial}{\partial x_1} q_1(X) + \frac{\partial}{\partial x_2} q_2(X) + \frac{\partial}{\partial x_3} q_3(X).$$

F-minimal surfaces

Of special interest are *anisotropic parametric functionals* of the form

$$\mathscr{F}[X] := \iint_B F(X_u \times X_v)\, dudv$$

with Lagrangians independent of the space point X. Critical points for such functionals are so-called *F-minimal surfaces.* Elaborating the imbedding of this class of immersions into the classical calculus of variations is one topic of the following considerations.

Critical points of $\mathscr{F}[X]$ are sort of analytical models for crystall growth processes motivated by the work of Wulff in [174]. Spherical surfaces of this kind are commonly called *Wulff surfaces.* We particularly refer the reader to Taylor [153], [154], or Morgan [120].

Our analysis follows the ideas of Sauvigny [141] in essential points.

11.2 Further regularity assumptions

We consider the general parametric variational problem

$$\mathscr{B}[X] = \iint_B F(X, X_u \times X_v)\, dudv \longrightarrow \text{extr!}$$

with a Lagrangian density

$$F \in C^5(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}, \mathbb{R}).$$

We assume that $F(X,Z)$ satisfies the following conditions.

(H) *Homogeneity:* For all $\lambda > 0$ it holds

$$F(X, \lambda Z) = \lambda F(X,Z).$$

(D) *Positive definiteness:* There exist two real constants $0 < m_1 \le m_2 < \infty$ with

$$m_1|Z| \le F(X,Z) \le m_2|Z|.$$

(E) *Ellipticity:* There exist two real constants $0 < M_1 \le M_2 < \infty$ such that

$$M_1|\xi|^2 \le \xi \circ \mathbf{F}_{ZZ}(X,Z) \circ \xi \le M_2|\xi|^2 \quad \text{for all } \xi \perp Z$$

with the matrix

$$\mathbf{F}_{ZZ}(X,Z) = (F_{z^i z^j}(X,Z))_{i,j=1,2,3} \in \mathbb{R}^{3\times 3}.$$

This latter ellipticity is equivalent to the *convexity condition*

$$F(X,\lambda Z_1+\mu Z_2)\le \lambda F(X,Z_1)+\mu F(X,Z_2)$$

for all real $\lambda,\mu \in > 0$ and all $Z_1,Z_2\in\mathbb{R}^3\setminus\{0\}$.

Definition 11.1. We additionally set

$$\begin{aligned}
F_X(X,Z) &:= \big(F_{x^1}(X,Z),F_{x^2}(X,Z),F_{x^3}(X,Z)\big)\in\mathbb{R}^3,\\
F_Z(X,Z) &:= \big(F_{z^1}(X,Z),F_{z^2}(X,Z),F_{z^3}(X,Z)\big)\in\mathbb{R}^3,\\
\mathbf{F}_{XX}(X,Z) &:= \big(F_{x^ix^j}(X,Z)\big)\in\mathbb{R}^{3\times 3},\\
\mathbf{F}_{XZ}(X,Z) &:= \big(F_{x^iz^j}(X,Z)\big)\in\mathbb{R}^{3\times 3}.
\end{aligned}$$

The Euler homogeneity conditions

Differentiation of the Lagrangian $F(X,Z)$ yields the so-called *Euler homogeneity conditions* as follows.

Lemma 11.1. *The vector $F_Z(X,Z)$ is positive homogeneous of degree 0 w.r.t. Z, the matrix $\mathbf{F}_{ZZ}(X,Z)$ is positive homogeneous of degree -1 w.r.t. Z. Furthermore there hold the relations*

$$\begin{aligned}
&F_Z(X,Z)\cdot Z=F(X,Z),\\
&\mathbf{F}_{ZZ}(X,Z)\circ Z=0,\\
&\mathbf{F}_{XZ}(X,Z)\circ Z=F_X(X,Z)
\end{aligned}$$

for all $X\in\mathbb{R}^3$ and all $Z\in\mathbb{R}^3\setminus\{0\}$.

Proof. Differentiation of the homogeneity condition (H) w.r.t. λ yields

$$F_Z(X,\lambda Z)\cdot Z=F(X,Z).$$

Thus the first statement follows if we set $\lambda=1$. A further differentiation yields $\mathbf{F}_{ZZ}(X,Z)\circ Z=0$. Computing finally the derivative of $F_X(X,\lambda Z)=\lambda F_X(X,Z)$ w.r.t. λ shows the third relation. The lemma is proved. $\square$

Parameter invariance

We show that the homogeneity condition ensures the parameter invariance of the functional $\mathscr{B}[X]$.

Proposition 11.1. *$\mathscr{B}[X]$ is invariant w.r.t. parameter transformations from class $\mathfrak{P}$ and does not depend on the chosen parameter domain iff it holds the homogeneity condition (H).*

Proof. 1. Let $\tilde{u} = \tilde{u}(u,v)$, $\tilde{v} = \tilde{v}(u,v)$ be a parameter transformation of class $\mathfrak{P}$. Assuming that (H) holds, we compute

$$\iint\limits_B F(X, X_u \times X_v)\, dudv = \iint\limits_B F(X, (\tilde{u}_u\tilde{v}_v - \tilde{u}_v\tilde{v}_u) X_{\tilde{u}} \times X_{\tilde{v}})\, dudv$$

$$= \iint\limits_B F(X, X_{\tilde{u}} \times X_{\tilde{v}})(\tilde{u}_u\tilde{v}_v - \tilde{u}_v\tilde{v}_u)\, dudv = \iint\limits_{\widetilde{B}} F(X, X_{\tilde{u}} \times X_{\tilde{v}})\, d\tilde{u}d\tilde{v}$$

which proves the parameter invariance of $\mathscr{B}[X]$.

2. The value of the functional $\mathscr{B}[X]$ does not depend on the parametrization. We set $\sigma := \tilde{u}_u\tilde{v}_v - \tilde{u}_v\tilde{v}_u$ and infer

$$\iint\limits_{\widetilde{B}} F(X, X_{\tilde{u}} \times X_{\tilde{v}})\, d\tilde{u}d\tilde{v} = \iint\limits_B F(X, \sigma^{-1} X_u \times X_v)\sigma\, dudv$$

$$\stackrel{!}{=} \iint\limits_B F(X, X_u \times X_v)\, dudv.$$

Applying this formula to the special transformation $\tilde{u} = \sqrt{\lambda}\, u$, $\tilde{v} = \sqrt{\lambda}\, v$ for real $\lambda > 0$ gives the homogeneity condition in integral form

$$\iint\limits_B F(X, \lambda X_u \times X_v)\, dudv = \iint\limits_B \lambda F(X, X_u \times X_v)\, dudv.$$

But this identity holds true on arbitrary discs $B_r(w)$, i.e.

$$\iint\limits_{B_r(w)} F(X, \lambda X_u \times X_v)\, dudv = \iint\limits_{B_r(w)} \lambda F(X, X_u \times X_v)\, dudv.$$

This proves the homogeneity condition (H) and thus the proposition. □

Condition (D) for compactly immersed surfaces

Consider an immersion $X\colon B \to \mathbb{R}^3$ satisfying

$$X(u,v) \in \mathscr{K} \subset \mathbb{R}^3 \quad \text{for all } (u,v) \in B$$

with a compact subset $\mathscr{K} \subset \mathbb{R}^3$. We want to verify condition (D). For this set

$$m_1 := \inf_{X \subset \mathscr{K}} \inf_{E \in S^2} F(X,E), \quad m_2 := \sup_{X \subset \mathscr{K}} \sup_{E \in S^2} |F_Z(X,E)|$$

with the unit sphere $S^2 := \{Z \in \mathbb{R}^3 : |Z| = 1\}$.

Then the homogeneity relation (H) implies

$$m_1|Z| \le F(X,Z) \le m_2|Z| \quad \text{for all } (X,Z) \in \mathscr{K} \times \mathbb{R}^3 \setminus \{0\}.$$

Namely, for fixed $Z \in \mathbb{R}^3 \setminus \{0\}$ and arbitrary $X \in \mathscr{K}$ we compute

$$\begin{aligned} F(X,Z) &\le \sup_{X\subset\mathscr{K}} F(X,Z) = \sup_{X\subset\mathscr{K}} F_Z(X,Z)\cdot Z \\ &\le \sup_{X\subset\mathscr{K}} |F_Z(X,Z)||Z| \le \sup_{X\subset\mathscr{K}} \sup_{E\in S^2} |F_Z(X,E)||Z| = m_2\,|Z| \end{aligned}$$

and analogously

$$m_1 \le F(X,Z/|Z|) = \frac{1}{|Z|}F(X,Z).$$

Thus we arrive at condition (D).

11.3 The first variation

The Euler-Lagrange equations

In this section we want to derive the Euler-Lagrange equations for critical points $X\colon B \to \mathbb{R}^3$ of the parametric functional $\mathscr{B}[X]$.

We use the following abbreviation for the triple product

$$[X,Y,Z] := X\cdot(Y\times Z), \quad X,Y,Z \in \mathbb{R}^3\,.$$

Theorem 11.1. *Let the immersion $X\colon B\to\mathbb{R}^3$ be critical for $\mathscr{B}[X]$. Then it satisfies*

$$[\mathbf{F}_{ZZ}(X,N)\circ N_u, N, X_v] + [\mathbf{F}_{ZZ}(X,N)\circ N_v, X_u, N] = W\,\mathrm{trace}\,\mathbf{F}_{XZ}(X,N) \quad \text{in } B \tag{11.1}$$

with the trace term

$$\mathrm{trace}\,\mathbf{F}_{XZ}(X,Z) = F_{x^1z^1}(X,Z) + F_{x^2z^2}(X,Z) + F_{x^3z^3}(X,Z).$$

Proof. Consider the variation

$$\widetilde{X}(u,v) := X(u,v) + \varepsilon\varphi(u,v)N(u,v), \quad (u,v)\in B,$$

with a test function $\varphi \in C_0^\infty(B,\mathbb{R})$ and $\varepsilon \in (-\varepsilon_0,+\varepsilon_0)$ small. First we have

$$\widetilde{X}_u \times \widetilde{X}_v = X_u\times X_v + \varepsilon\,(X_u\times N_v + N_u\times X_v)\varphi + \varepsilon\,(\varphi_v X_u\times N + \varphi_u N\times X_v) + o(\varepsilon).$$

We compute

$$\begin{aligned}\frac{\partial}{\partial \varepsilon}\mathscr{B}[\widetilde{X}] &= \frac{\partial}{\partial \varepsilon}\iint_B F(\widetilde{X},\widetilde{X}_u \times \widetilde{X}_v)\,dudv\\ &= \iint_B F_X(\widetilde{X},\widetilde{X}_u \times \widetilde{X}_v)\cdot \frac{\partial}{\partial \varepsilon}\widetilde{X}\,dudv\\ &\quad + \iint_B F_Z(\widetilde{X},\widetilde{X}_u \times \widetilde{X}_v)\cdot \frac{\partial}{\partial \varepsilon}(\widetilde{X}_u \times \widetilde{X}_v)\,dudv.\end{aligned}$$

Let as usual

$$\delta\mathscr{B}[X] = \frac{\partial}{\partial \varepsilon}\mathscr{B}[\widetilde{X}]\big|_{\varepsilon=0}\,.$$

Using $F_Z(X,\lambda Z) = F_Z(X,Z)$ we arrive at

$$\begin{aligned}\delta\mathscr{B}[X] &= \iint_B F_X(X,X_u \times X_v)\cdot N\varphi\,dudv\\ &\quad + \iint_B F_Z(X,X_u \times X_v)\cdot (X_u \times N_v + N_u \times X_v)\varphi\,dudv\\ &\quad + \iint_B F_Z(X,X_u \times X_v)\cdot (X_u \times N\varphi_v + N \times X_v\varphi_u)\,dudv\\ &= \iint_B [F_X(X,N),X_u,X_v]\varphi\,dudv\\ &\quad + \iint_B \Big\{[F_Z(X,N),X_u,N_v] + [F_Z(X,N),N_u,X_v]\Big\}\varphi\,dudv\\ &\quad + \iint_B \Big\{[F_Z(X,N),X_u,N]\varphi_v + [F_Z(X,N),N,X_v]\varphi_u\Big\}\,dudv.\end{aligned}$$

The last two integrals are equal to

$$\begin{aligned}&\iint_B \operatorname{div}\big([F_Z(X,N),N,X_v]\varphi, [F_Z(X,N),X_u,N]\varphi\big)\,dudv\\ &\quad - \iint_B \Big\{[\mathbf{F}_{ZX}(X,N)\circ X_u,N,X_v] + [\mathbf{F}_{ZX}(X,N)\circ X_v,X_u,N]\Big\}\varphi\,dudv\\ &\quad - \iint_B \Big\{[\mathbf{F}_{ZZ}(X,N)\circ N_u,N,X_v] + [\mathbf{F}_{ZZ}(X,N)\circ N_v,X_u,N]\Big\}\varphi\,dudv.\end{aligned}$$

The integral over the divergence term vanishes due to $\varphi|_{\partial B} = 0$, i.e.

$$\iint\limits_B \operatorname{div}\left([F_Z(X,N),N,X_v]\varphi,[F_Z(X,N),X_u,N]\varphi\right)du\,dv = 0.$$

Furthermore, $F_X(X,Z) = \mathbf{F}_{XZ}(X,Z)\circ Z$ yields

$$\begin{aligned}[F_X(X,N),X_u,X_v] &= [\mathbf{F}_{XZ}(X,N)\circ N,X_u,X_v] = [X_u,X_v,\mathbf{F}_{XZ}(X,N)\circ N] \\ &= [X_u,X_v,\mathbf{F}_{ZX}(X,N)\circ N]\end{aligned}$$

where we make use of

$$Z\circ\mathbf{F}_{XZ}(X,Z)\circ Z = Z\circ\mathbf{F}_{ZX}(X,Z)\circ Z.$$

Now we insert

$$\begin{aligned}&[\mathbf{F}_{ZX}(X,N)\circ X_u,X_v,N] + [X_u,\mathbf{F}_{ZX}(X,N)\circ X_v,N] + [X_u,X_v,\mathbf{F}_{ZX}(X,N)\circ N] \\ &\quad = [X_u,X_v,N]\operatorname{trace}\mathbf{F}_{ZX}(X,N) = W\operatorname{trace}\mathbf{F}_{XZ}(X,N).\end{aligned}$$

into the $\delta\mathscr{B}[X]$ to get

$$\begin{aligned}\delta\mathscr{B}[X] &= \iint\limits_B [F_X(X,N),X_u,X_v]\varphi\,du\,dv \\ &\quad - \iint\limits_B \Big\{[\mathbf{F}_{ZX}(X,N)\circ X_u,N,X_v] + [\mathbf{F}_{ZX}(X,N)\circ X_v,X_u,N]\Big\}\varphi\,du\,dv \\ &\quad - \iint\limits_B \Big\{[\mathbf{F}_{ZZ}(X,N)\circ N_u,N,X_v] + [\mathbf{F}_{ZZ}(X,N)\circ N_v,X_u,N]\Big\}\varphi\,du\,dv \\ &= \iint\limits_B \operatorname{trace}\mathbf{F}_{XZ}(X,N)W\varphi\,du\,dv \\ &\quad - \iint\limits_B \Big\{[\mathbf{F}_{ZZ}(X,N)\circ N_u,N,X_v] + [\mathbf{F}_{ZZ}(X,N)\circ N_v,X_u,N]\Big\}\varphi\,du\,dv.\end{aligned}$$

The fundamental lemma of the calculus of variations yields

$$[\mathbf{F}_{ZZ}(X,N)\circ N_u,N,X_v] + [\mathbf{F}_{ZZ}(X,N)\circ N_v,X_u,N] = W\operatorname{trace}\mathbf{F}_{XZ}(X,N).$$

The theorem is proved. $\square$

Critical points are immersions of mean curvature type

Following Sauvigny [141] we introduce a special weight matrix $\mathbf{W}(X,Z)$ to transform the Euler-Lagrange equations from the previous theorem into the so-called system of mean-curvature type.

Theorem 11.2. *A critical point $X\colon B \to \mathbb{R}^3$ of $\mathscr{B}[X]$ solves the nonlinear elliptic system*

$$\nabla_{ds_W^2}(X,N) := \sum_{i,j=1}^{2} h^{ij} X_{u^i} \cdot N_{u^j} = -2H_W(X,N) \quad \text{in } B \tag{11.2}$$

with the parameter invariant Beltrami operator $\nabla_{ds_W^2}(\cdot,\cdot)$ w.r.t.

- *the metric*

$$ds_W^2 := \sum_{i,j=1}^{2} h_{ij}\, du^i du^j\,, \quad h_{ij} := X_{u^i} \circ \mathbf{W}(X,N) \circ X_{u^j}\,, \tag{11.3}$$

- *the weight matrix*

$$\mathbf{W}(X,Z) := \left(\frac{1}{\sqrt{\det \mathbf{F}_{ZZ}(X,Z)}} \mathbf{F}_{ZZ}(X,Z) + (z^i z^j)_{i,j=1,2,3} \right)^{-1} \tag{11.4}$$

  for $X \in \mathbb{R}^3$ and $Z \in S^2$,
- *and the weighted mean curvature*

$$H_W(X,Z) = \frac{1}{2\sqrt{\det \mathbf{F}_{ZZ}(X,N)}} \operatorname{trace} \mathbf{F}_{XZ}(X,N).$$

Before we come to the proof of this theorem we want to verify first that $\mathbf{W}(X,N)$ from this definition actually represents a weight matrix in the sense of our definition from chapter 4. Then we come to the proof the parameter invariance of the Beltrami operator $\nabla_{ds_W^2}(\varphi,\psi)$.

Properties of the weight matrix

In chapter 4 we introduced weight matrices $\mathbf{W}(X,Z)$ on the tangential space of surfaces with arbitrary codimensions. The properties (W1) to (W4) given there reduce to the following:

For all $X \in \mathbb{R}^3$ and all $Z \in \mathbb{R}^3 \setminus \{0\}$ there hold

(W1) $\mathbf{W}(X,Z)$ is invariant w.r.t. the choice of the normal vector, i.e.

$$\mathbf{W}(X,\lambda Z) = \mathbf{W}(X,Z);$$

(W2) $\mathbf{W}(X,Z)$ acts non-trivially on the tangent space, i.e.

$$\mathbf{W}(X,Z)\big|_{\mathfrak{T}_X(w)} : \mathfrak{T}_X(w) \longrightarrow \mathfrak{T}_X(w), \quad \text{in particular,}$$
$$\operatorname{rank}\mathbf{W}(X,Z)\big|_{\mathfrak{T}_X(w)} = 2, \quad \mathbf{W}(X,Z)\circ Z = Z;$$

(W3) $\mathbf{W}(X,Z)$ is positive definite, i.e. with a real constant $\omega_0 \in [0,\infty)$ we have

$$(1+\omega_0)^{-1}|\xi|^2 \le \xi \circ \mathbf{W}(X,Z) \circ \xi \le (1+\omega_0)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3;$$

(W4) $\mathbf{W}(X,Z)$ is normalized in the following sense

$$\det \mathbf{W}(X,Z) = 1.$$

Let us now consider the special weight matrix $\mathbf{W}(X,Z)$ from the previous theorem. We show that it satisfies these properties (W1) to (W4).

1. First we continue $\mathbf{W}(X,Z)$ constantly along lines $\{\lambda Z\}$ for $\lambda \in \mathbb{R}$ and $Z \in S^2$ such that it holds

$$\mathbf{W}(X,\lambda Z) = \mathbf{W}(X,Z) \quad \text{for all } \lambda \in \mathbb{R}.$$

   This proves (W1).

2. The weight matrix $\mathbf{W}(X,Z)$ consists of the mapping $\mathbf{F}_{ZZ}(X,Z)$ which acts on the tangential space, and of the mapping $(z^i z^j)_{i,j=1,2,3}$. There hold

$$\operatorname{rank}\mathbf{F}_{ZZ}(X,Z) = 2, \quad \operatorname{rank}(z^i z^j)_{i,j=1,2,3} = 1.$$

   The weight matrix is homogeneous of degree 0 in Z. We have

$$\mathbf{W}(X,Z)\circ Z = Z \quad \text{for all } Z \in S^2$$

   where we note

$$\mathbf{F}_{ZZ}(X,Z)\circ Z = 0$$

   as well as

$$(z^i z^j)_{i,j=1,2,3}\circ Z = \begin{pmatrix} (z^1)^3 + z^1(z^2)^2 + z^1(z^3)^2 \\ (z^1)^2 z^2 + (z^2)^3 + z^2(z^3)^2 \\ (z^1)^2 z^3 + (z^2)^2 z^3 + (z^3)^2 \end{pmatrix} = Z$$

   for all $Z \in S^2$. We infer (W4) from

$$\det \mathbf{W}(X,Z) = 1 \quad \text{for all } X \in \mathbb{R}^3 \text{ and } Z \in S^2.$$

Property (W2) follows now from

$$\mathbf{W}(X,Z)^{-1}\Big|_{\mathfrak{T}_Z} = \frac{1}{\sqrt{\det \mathbf{F}_{ZZ}(X,Z)}}\,\mathbf{F}_{ZZ}(X,Z) : \mathfrak{T}_Z \longrightarrow \mathfrak{T}_Z \quad \text{for all } Z \in S^2 .$$

3. Let finally $\xi \perp N$ be an arbitrary tangential vector. Then the ellipticity condition (E) on $\mathbf{F}_{ZZ}(X,Z)$ ensures

$$\xi \circ \mathbf{W}(X,N)^{-1} \circ \xi = \frac{1}{\sqrt{\det \mathbf{F}_{ZZ}(X,N)}}\,\xi \circ \mathbf{F}_{ZZ}(X,N) \circ \xi \le \frac{M_2}{M_1}\,|\xi|^2$$

as well as

$$\xi \circ \mathbf{W}(X,N)^{-1} \circ \xi \ge \frac{M_1}{M_2}\,|\xi|^2 .$$

Thus (W3) follows with the setting

$$\omega_0 := \frac{M_2}{M_1} - 1.$$

Note that ω_0 *depends only on the ratio of the eigenvalue bounds on* $\mathbf{F}_{ZZ}(X,Z)$.

The invariant Beltrami operator

We come back to the operator $\nabla_{ds^2_W}(\varphi,\psi)$, see e.g. Blaschke and Leichtweiß [15].

Definition 11.2. The *first Beltrami operator* for continuously differentiable functions $\varphi,\psi\colon B \to \mathbb{R}$ w.r.t. the line element ds^2_W is defined as

$$\nabla_{ds^2_W}(\varphi,\psi) := \sum_{i,j=1}^{2} h^{ij}\varphi_{u^i}\psi_{u^j} .$$

Analogously we define the first Beltrami operator w.r.t. any other Riemannian line element ds^2, or even for smooth vector-valued mappings X and Y, namely

$$\nabla_{ds^2_W}(X,Y) = \sum_{i,j=1}^{2} h^{ij} X_{u^i}\cdot X_{u^j} .$$

To prove its parameter invariance, let $u^1(v^\alpha) \in \mathfrak{P}$ be parameter transformation from class $\mathfrak{P}$. Then

$$\begin{aligned}\sum_{i,j=1}^{2} h^{ij}L_{ij} &= \sum_{i,j=1}^{2}\sum_{\kappa,\lambda=1}^{2}\sum_{\mu,\nu=1}^{2} \overline{\Lambda}{}^i_\kappa \overline{\Lambda}{}^j_\lambda \Lambda^\mu_i \Lambda^\nu_j h^{\kappa\lambda} L_{\mu\nu} = \sum_{\kappa,\lambda=1}^{2}\sum_{\mu,\nu=1}^{2} \delta^\mu_\kappa \delta^\nu_\lambda h^{\kappa\lambda} L_{\mu\nu} \\ &= \sum_{\kappa,\lambda=1}^{2} h^{\kappa\lambda} L_{\kappa\lambda} .\end{aligned}$$

Proof of the mean curvature type representation

Now we come to the proof of Theorem 11.2. Together with lemma 5.3, i.e.

$$(\mathbf{M}\circ X)\times(\mathbf{M}\circ Y)=(\det\mathbf{M})\mathbf{M}^{-1}\circ(X\times Y)$$

for all non-singular and symmetric matrices $\mathbf{M}\in\mathbb{R}^{3\times 3}$ and all $X,Y\in\mathbb{R}^3$, we infer

$$\begin{aligned}
&\left\{\mathbf{W}(X,N)^{-\frac{1}{2}}\circ N_u\right\}\times\left\{\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v\right\}\\
&\quad=\left\{\mathbf{W}(X,N)^{\frac{1}{2}}\circ\mathbf{W}(X,N)^{-1}\circ N_u\right\}\times\left\{\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v\right\}\\
&\quad=\mathbf{W}(X,N)^{-\frac{1}{2}}\circ\left\{(\mathbf{W}(X,N)^{-1}\circ N_u)\times X_v\right\}\\
&\quad=(\mathbf{W}(X,N)^{-1}\circ N_u)\times X_v\,.
\end{aligned}$$

Analogously we obtain

$$\left\{\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u\right\}\times\left\{\mathbf{W}(X,N)^{-\frac{1}{2}}\circ N_v\right\}=X_u\times(\mathbf{W}(X,N)^{-1}\circ N_u).$$

Now insert the definitions of the weighted mean curvature $H_G(X,Z)$ and the weight matrix $\mathbf{W}(X,Z)$ from the theorem into the Euler-Lagrange system to get

$$\begin{aligned}
-2H_W(X,N)W&=\frac{1}{\sqrt{\det\mathbf{F}_{ZZ}(X,N)}}\,[X_u,\mathbf{F}_{ZZ}(X,N)\circ N_v,N]\\
&\quad+\frac{1}{\sqrt{\det\mathbf{F}_{ZZ}(X,N)}}\,[\mathbf{F}_{ZZ}(X,N)\circ N_u,X_v,N]\\
&=[X_u,\mathbf{W}(X,N)^{-1}\circ N_v,N]+[\mathbf{W}(X,N)^{-1}\circ N_u,X_v,N]\\
&=[\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u,\mathbf{W}(X,N)^{-\frac{1}{2}}\circ N_v,N]\\
&\quad+[\mathbf{W}(X,N)^{-\frac{1}{2}}\circ N_u,\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v,N].
\end{aligned}$$

Multiplication by

$$[N,\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u,\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v]=[N,X_u,X_v]=W$$

yields

$$-2H_W(X,N)W^2=\begin{vmatrix}N_u\cdot X_u & X_u\circ\mathbf{W}(X,N)\circ X_v & 0\\ N_u\cdot X_v & X_v\circ\mathbf{W}(X,N)\circ X_v & 0\\ 0 & 0 & 1\end{vmatrix}+\begin{vmatrix}X_u\circ\mathbf{W}(X,N)\circ X_u & N_v\cdot X_u & 0\\ X_v\circ\mathbf{W}(X,N)\circ X_u & N_v\cdot X_v & 0\\ 0 & 0 & 1\end{vmatrix}.$$

This is equivalent to

$$\begin{aligned}-2H_W(X,N)W^2 &= h_{22}(X_u\cdot N_u)-2h_{12}(X_v\cdot N_u)+h_{11}(X_v\cdot N_v)\\ &= W^2\sum_{i,j=1}^{2}h^{ij}(X_{u^i}\cdot N_{u^j}).\end{aligned}$$

This proves the stated representation. □

Example: Minimal surfaces

Consider again the area function

$$\mathscr{A}[X]=\iint\limits_B |X_u\times X_v|\,dudv$$

with the Lagrangian $F(Z)=|Z|$. There hold

$$F_Z(X,Z)=\frac{Z}{|Z|}\,,\quad F_X(X,Z)\equiv 0$$

as well as

$$\mathbf{F}_{ZZ}(X,Z)=\left(\frac{\delta_{ij}}{|Z|}-\frac{z^iz^j}{|Z|^3}\right)_{i,j=1,2,3}$$

with the Kronecker symbol δ_{ij}, thus $\mathbf{W}(X,Z)\equiv\mathbb{E}^3$ with the three-dimensional unit matrix $\mathbb{E}^3$. *Thus minimal surfaces $X\colon B\to\mathbb{R}^3$ as critical points of the area functional $\mathscr{A}[X]$ fulfill*

$$\nabla_{ds^2}(X,N)=\frac{g_{22}L_{11}-2g_{12}L_{12}+g_{11}L_{22}}{W^2}=0\quad\text{in } B$$

with the classical metric

$$ds^2=g_{11}\,du+2g_{12}\,dudv+g_{22}\,dv^2\,.$$

It particularly holds $H\equiv 0$ for its scalar mean curvature.

Example: Immersions with prescribed mean curvature

We consider two variational problems from the introduction with critical points possessing either constant mean curvature or mean curvature represented by the divergence of the discussed vector field in space.

In both examples the weight matrix $\mathbf{W}(X,Z)$ simply equals the three-dimensional unit matrix $\mathbb{E}^3$.

1. *Immersions with constant mean curvature*
   First consider the functional
$$\iint\limits_B \left\{ |X_u \times X_v| + \frac{2h_0}{3} X \cdot (X_u \times X_v) \right\} dudv.$$
   We immediately compute
$$F_X(X,Z) = \frac{2h_0}{3} Z, \quad F_Z(X,Z) = \frac{Z}{|Z|} + \frac{2h_0}{3} X,$$
   furthermore
$$\mathbf{F}_{XX}(X,Z) = \mathbf{0}, \quad \mathbf{F}_{XZ}(X,Z) = \frac{2h_0}{3} \mathbb{E}^3, \quad \mathbf{F}_{ZZ}(X,Z) = \left(\frac{\delta_{ij}}{|Z|} - \frac{z^i z^j}{|Z|^3} \right)_{i,j=1,2,3}$$
   as well as
$$\mathbf{F}_{XZ}(X,Z) = \mathbf{F}_{ZX}(X,Z).$$
   Thus cricital points $X\colon B \to \mathbb{R}^3$ *of this functional have constant mean curvature*
$$H(u,v) \equiv h_0 \quad \text{in } B,$$
   and they solve
$$\nabla_{ds^2}(X,N) = -2h_0$$
   w.r.t. the non-weighted line element ds^2.

2. *Immersions with prescribed mean curvature*
   Now we come to Hildebrandt's functional
$$\iint\limits_B \left\{ |X_u \times X_v| + 2Q(X) \cdot (X_u \times X_v) \right\} dudv$$
   with a prescribed vector field $Q(X) = (q_1(X), q_2(X), q_3(X))$. We have
$$\mathbf{F}_{XZ}(X) = (F_{x^i z^j})_{i,j=1,2,3} = (2q_{j,x^i})_{i,j=1,2,3} \in \mathbb{R}^{3\times 3}.$$
   Thus critical points $X\colon B \to \mathbb{R}^3$ *possess the mean curvature*
$$H(X) = \sum_{i=1}^{3} q_{i,x^i}(X) = \operatorname{div}_X Q(X) \quad \text{in } B$$
   with the spatial divergence operator div_X. It holds
$$\nabla_{ds^2}(X,N) = -2\operatorname{div}_X Q(X)$$
   again w.r.t. the classical metric ds^2.

Example: Immersions of minimal surface type. F-minimal surfaces

For critical points of the anisotropic parametric functional

$$\mathscr{F}[X] = \iint_B F(X_u \times X_v)\,dudv$$

we immediately infer

$$\mathbf{F}_{XZ}(X,Z) \equiv 0.$$

Now introducing the weight matrix $\mathbf{W}(X,Z)$ from Theorem 11.2 we arrive at

$$\nabla_{ds_W^2}(X,N) = 0 \quad \text{in } B$$

with the weighted line element

$$ds_W^2 = h_{11}\,du + 2h_{12}\,dudv + h_{22}\,dv^2$$

from (11.3). *Thus critical points of $\mathscr{F}[X]$ are characterized by the property that their weighted mean curvature H_W from Theorem 11.2 vanishes identically,*

$$H_W \equiv 0 \quad \text{in } B.$$

These immersions of minimal surface type are called *F-minimal surfaces.*

Example: Immersions with prescribed weighted mean curvature

We consider again two examples.

1. *Immersions of constant mean curvature*
   First let
   $$\iint_B \left\{ F(X_u \times X_v) + \frac{2h_0}{3}\,X \cdot (X_u \times X_v) \right\} dudv$$
   with a real constant $h_0 \in \mathbb{R}$. It generalizes the variational problem for surfaces with constant mean curvature by substituting $|Z|$ with a mapping $F(Z)$. Now let us denote the integrand by $\widetilde{F}(u,v)$. Then we calculate
   $$\widetilde{F}_X(X,Z) = \frac{2h_0}{3}\,Z, \quad \widetilde{F}_Z(X,Z) = F_Z(Z) + \frac{2h_0}{3}\,X$$
   as well as
   $$\widetilde{\mathbf{F}}_{XX}(X,Z) = \mathbf{0}, \quad \widetilde{\mathbf{F}}_{XZ}(X,Z) = \frac{2h_0}{3}\,\mathbb{E}^3\,, \quad \widetilde{\mathbf{F}}_{ZZ}(X,Z) = \mathbf{F}_{ZZ}(Z).$$

Thus critical points $X: B \to \mathbb{R}^3$ have weighted mean curvature

$$H_W(N) = \frac{h_0}{\sqrt{\det \mathbf{F}_{ZZ}(N)}} \in \left[\frac{h_0}{M_2}, \frac{h_0}{M_1}\right]$$

w.r.t. the weight matrix $\mathbf{W}(Z)$ from Theorem 11.2. In particular, they satisfy

$$\nabla_{ds_W^2}(X,N) = -\frac{2h_0}{\sqrt{\det \mathbf{F}_{ZZ}(N)}} \quad \text{in } B.$$

2. *Immersions of prescribed mean curvature type*
   In generalization, we now consider the Lagrangian density

$$\widetilde{F}(X, X_u \times X_v) = F(X_u \times X_v) + Q(X) \cdot (X_u \times X_v).$$

   Analogously to the foregoing calculations we compute

$$H_W(X,N) = \frac{1}{\sqrt{\det \mathbf{F}_{ZZ}(N)}} \operatorname{div}_X Q(X)$$

   for the weighted mean curvature w.r.t. the above weight matrix $\mathbf{W}(Z) \in \mathbb{R}^{3\times 3}$ for cricital points $X: B \to \mathbb{R}^3$ of the associated variational problem.

11.4 Principal curvatures and weighted mean curvature

From the elliptic mean curvature type system

$$\nabla_{ds_W^2}(X,N) = -2H_W(X,N) \quad \text{in } B$$

w.r.t. some weight matrix $\mathbf{W}(X,Z)$ we want to deduce a relation between the weighted mean curvature $H_W(X,Z)$ and the principal curvatures κ_1 and κ_2 of an immersion X.

In the non-weighted case these curvatures satisfy

$$\kappa_1 + \kappa_2 = 2H(X,N)$$

with a prescribed mean curvature $H(X,Z)$.

In the general case it turns that out that the principal curvatures have to be multiplied with special weight factors to suffice a curvature relation like this.

A mean curvature relation

Our next result *does not only apply* to critical points of variational problems. Rather we formulate it for any immersion satisfying the above differential equation.

Theorem 11.3. *Let the immersion $X\colon B \to \mathbb{R}^3$ solve the nonlinear elliptic mean curvature type system*

$$\nabla_{ds^2_W}(X,N) = -2H_W(X,N)$$

with prescribed weighted mean curvature

$$H_W\colon \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}$$

w.r.t. a weight matrix $\mathbf{W}(X,Z)$. Consider a point $w \in B$, and let $Z_1(w)$ and $Z_2(w)$ denote the principal curvature directions of the surface X at this point. Set

$$\rho_1(X,N) := Z_1(w) \circ \mathbf{W}(X,N) \circ Z_1(w)\,,$$

$$\rho_2(X,N) := Z_2(w) \circ \mathbf{W}(X,N) \circ Z_2(w)$$

using principal curvature parameters around this point. Then it holds

$$\rho_1(X,N)\kappa_1(w) + \rho_2(X,N)\kappa_2(w) = 2H_W(X,N) \quad \textit{for all } w \in B. \tag{11.5}$$

On the principal curvatures

Before we come to the proof of this result we want to recall some important properties of principal curvature parametrizations.

For this purpose consider a point $(u,v) \in \mathring{B}$ and the circle $S^1(X(u,v))$ of radius 1 and with center $X(u,v)$ on the tangential plane $\mathfrak{T}_X(u,v)$.

Definition 11.3. The vector $Z \in S^1(X(u,v))$ is called a *principal curvature direction* at $X(u,v)$ if the mapping

$$Z = \partial X(u,v) \circ \begin{pmatrix} \dfrac{du}{dt} \\ \dfrac{dv}{dt} \end{pmatrix} \longmapsto L_{11}\left(\frac{du}{dt}\right)^2 + 2L_{12}\,\frac{du}{dt}\,\frac{dv}{dt} + L_{22}\left(\frac{dv}{dt}\right)^2$$

takes a stationary value. The smooth curve $c(t) := X(u(t),v(t))$ is called a *principal curvature line* if it holds

$$\frac{d}{dt}\,c(t) \neq 0$$

and if

$$\left|\frac{d}{dt}\,c(t)\right|^{-1}\,\frac{d}{dt}\,c(t)$$

represents a principal curvature direction.

A *plane normal section* of the surface at $X(w_0)$ along the tangential line of a curve $c = c(t)$ on that surface, attached at the point $X(w_0)$, is the projection of this curve onto the intersecting plane. The curvature of this new plane curve is called *normal curvature* at $X(w_0)$.

There are always two linearly independent principal curvature directions where the normal curvature takes its maximal and minimal value on the compact set $S^1(X(w_0))$. These quantities are exactly the principal curvatures.

Now let $\kappa_1(u,v) \neq \kappa_2(u,v)$ at some $(u,v) \in B$. Following Klingenberg [106], Theorem 3.6.6, we can introduce new parameters in the neighborhood of $(u,v) \in B$ such that the new parameter lines agree with the principal curvature directions there.

Proposition 11.2. *Under these conditions there hold the equations of Rodrigues*

$$N_u(u,v) = -\kappa_1(u,v)X_u(u,v), \quad N_v(u,v) = -\kappa_2(u,v)X_v(u,v) \tag{11.6}$$

as well as

$$X_u(u,v) \cdot N_v(u,v) = 0, \quad X_v(u,v) \cdot N_u(u,v) = 0.$$

In particular, the second fundamental form is diagonalised.

Now let $\kappa_1(u,v) = \kappa_2(u,v)$ for a point $(u,v) \in B$. Then all directions on the surface, attached at that point, are principal curvature directions since there is only one normal curvature. Such points are called *umbilical points.*

The next proposition can be found in Laugwitz [111].

Proposition 11.3. *At each point there are two linearly independent and orthogonal principal curvature directions where the normal curvatures take their extremal values.*

If the parameter system is chosen such that the parameter lines touch the principal curvature lines at some point $X(u,v)$ then there hold

$$L_{11}(u,v) = \kappa_1(u,v)g_{11}(u,v), \quad L_{22}(u,v) = \kappa_2(u,v)g_{22}(u,v) \tag{11.7}$$

as well as

$$g_{12}(u,v) = 0, \quad L_{12}(u,v) = 0 \tag{11.8}$$

with the coefficients g_{ij} and L_{ij} of the first resp. second fundamental form.

Proof of the mean curvature relation

Let us now come to the proof of the foregoing theorem.

Proof. Start with choosing principal curvature parameters (u,v) locally. We rewrite the mean curvature type system

$$\nabla_{ds_W^2}(X,N) = -2H_W(X,N) \quad \text{in } B$$

into the form

$$\sum_{i,j=1}^{2} L_{ij}(u,v)h^{ij}(u,v) = 2H_W(u,v) \quad \text{in } B,$$

$h_{ij} = X_{u^i} \circ \mathbf{W}(X,N) \circ X_{u^j}$ the coefficients of the weighted first fundamental form.

Then using (11.7) and (11.8) we find

$$\begin{aligned}
h^{11}L_{11}+h^{22}L_{22} &= -\frac{X_v\circ\mathbf{W}(X,N)\circ X_v}{W^2}\,(N_u\cdot X_u)-\frac{X_u\circ\mathbf{W}(X,N)\circ X_u}{W^2}\,(N_v\cdot X_v)\\
&= \frac{X_v\circ\mathbf{W}(X,N)\circ X_v}{W^2}\,|X_u|^2\kappa_1+\frac{X_u\circ\mathbf{W}(X,N)\circ X_u}{W^2}\,|X_v|^2\kappa_2\\
&= \left(\frac{|X_u|}{W}\,X_v\right)\circ\mathbf{W}(X,N)\circ\left(\frac{|X_u|}{W}\,X_v\right)\kappa_1\\
&\quad+\left(\frac{|X_v|}{W}\,X_u\right)\circ\mathbf{W}(X,N)\circ\left(\frac{|X_v|}{W}\,X_u\right)\kappa_2
\end{aligned}$$

with the area element $W=|X_u||X_v|$. If follows

$$\begin{aligned}
h^{11}L_{11}+h^{22}L_{22} &= (|X_v|^{-1}X_v)\circ\mathbf{W}(X,N)\circ(|X_v|^{-1}X_v)\,\kappa_1\\
&\quad+(|X_u|^{-1}X_u)\circ\mathbf{W}(X,N)\circ(|X_u|^{-1}X_u)\,\kappa_2\\
&=: (Z_1\circ\mathbf{W}(X,N)\circ Z_1)\,\kappa_1+(Z_2\circ\mathbf{W}(X,N)\circ Z_2)\,\kappa_2\\
&= \rho_1(X,N)\kappa_1+\rho_2(X,N)\kappa_2
\end{aligned}$$

with the orthonormal principal curvature directions

$$Z_1:=\frac{1}{|X_v|}\,X_v\,,\quad Z_2:=\frac{1}{|X_u|}\,X_u$$

and the weight factors $\rho_1(X,N)$ and $\rho_2(X,N)$ from the theorem. We arrive at

$$\nabla_{ds_W^2}(X,N)=-\sum_{i,j=1}^{2}h^{ij}L_{ij}=-\rho_1(X,N)\kappa_1-\rho_2(X,N)\kappa_2=-2H_W(X,N)$$

proving the statement. □

Remarks on the weight factors ρ_1 and ρ_2

Together with (W3) from chapter 4 we have

$$\frac{1}{1+\omega_0}\le\rho_1(X,N),\rho_2(X,N)\le 1+\omega_0\,.$$

In the non-weighted case $\mathbf{W}(X,Z)\equiv\mathbb{E}^3$ we particularly infer

$$\rho_1(X,Z)\equiv 1,\quad \rho_2(X,Z)\equiv 1$$

and therefore $\kappa_1+\kappa_2=2H$ with the classical scalar mean curvature.

We also infer

$$\kappa_1(u,v)\kappa_2(u,v) = K(u,v) \leq 0 \quad \text{in } B$$

for the Gaussian curvature K of a weighted minimal surface satisfying $H_W \equiv 0$.

We want to mention that the curvature relation

$$\rho_1(X,N)\kappa_1 + \rho_2(X,N)\kappa_2 = 0 \quad \text{in } B$$

for F-minimal surfaces critical for the anisotropic but homogeneous functional

$$\iint\limits_B F(X_u \times X_v)\, dudv$$

can already be found in Sauvigny [141]. From there we also took the methods presented before for deriving the general identities including also inhomogeneous variational problems.

11.5 Non-parametric differential equations

Nonparametric variational problems

In section 9.2 we have already computet the Euler-Lagrange equations

$$\frac{dF_{p_\sigma}(x,y,\zeta,\nabla\zeta)}{dx} + \frac{dF_{q_\sigma}(x,y,\zeta,\nabla\zeta)}{dy} = F_{z_\sigma}(x,y,\zeta,\nabla\zeta),$$

where $\sigma = 1,\ldots,n$, for non-parametric functionals of the form

$$\mathscr{F}[\zeta] = \iint\limits_\Omega F(x,y,\zeta,\nabla\zeta)\, dxdy$$

witht the setting

$$\zeta = (\zeta_1,\ldots,\zeta_n).$$

If F equals the area integrand, i.e.

$$\mathscr{A}[\zeta] = \iint\limits_\Omega \sqrt{1+p^2+q^2+p^2q^2-(p\cdot q)^2}\, dxdy$$

with $p := \zeta_x$, $q := \zeta_y$, we arrive at the non-parametric minimal surface system

$$\operatorname{div}\frac{(p_\sigma,q_\sigma)}{W} = -\operatorname{div}\frac{\big(p_\sigma|q|^2 - q_\sigma(p\cdot q), q_\sigma|p|^2 - p_\sigma(p\cdot q)\big)}{W} \quad \textit{in } \Omega$$

for $\sigma = 1,\ldots,n$.

The right hand side in this system vanishes identically in case $n = 1$ and gives us the non-parametric minimal surface equation in one codimension

$$\operatorname{div} \frac{\nabla \zeta}{W} = 0 \quad \text{in } \Omega .$$

Let us consider the more general integrand

$$F(x,y,z,p,q) = \Gamma(x,y,z)\sqrt{1+p^2+q^2+p^2q^2-(p\cdot q)^2}$$

where $z = (z_1,\dots,z_n)$ etc.

Corollary 11.1. *Critical points $(x,y,\zeta_1,\dots,\zeta_n)$ of the functional*

$$\iint_\Omega \Gamma(x,y,\zeta)W\,dxdy$$

satisfy the Euler-Lagrange system

$$\begin{aligned}
\operatorname{div} \frac{(p_\sigma,q_\sigma)}{W} &= 2H(X,N_\sigma)\,\frac{\sqrt{1+p_\sigma^2+q_\sigma^2}}{W} \\
&+ \frac{1}{\Gamma W}\left\{\left[p^2+q^2+p^2q^2-(p\cdot q)^2\right]\Gamma_{z_\sigma} - \sum_{\omega=1}^{n}(p_\sigma p_\omega + q_\sigma q_\omega)\Gamma_{z_\omega}\right\} \\
&- \frac{1}{\Gamma}\operatorname{div}\left(\frac{p_\sigma q^2 - q_\sigma(p\cdot q)}{W}\,\Gamma, \frac{q_\sigma p^2 - p_\sigma(p\cdot q)}{W}\,\Gamma\right)
\end{aligned}$$

in Ω for $\sigma = 1,\dots,n$, with the mean curvature field

$$H(X,N_\sigma) := \frac{\Gamma_X(x,y,z)\cdot N_\sigma}{2\Gamma(x,y,z)}$$

w.r.t. to the Euler unit normal vectors

$$\begin{aligned}
N_1 &:= \frac{1}{\sqrt{1+|\nabla\zeta_1|^2}}\,(-\zeta_{1,x},-\zeta_{1,y},1,0,0,\dots,0,0),\\
N_2 &:= \frac{1}{\sqrt{1+|\nabla\zeta_2|^2}}\,(-\zeta_{2,x},-\zeta_{2,y},0,1,0,\dots,0,0),\\
&\;\;\vdots\\
N_n &:= \frac{1}{\sqrt{1+|\nabla\zeta_n|^2}}\,(-\zeta_{n,x},-\zeta_{n,y},0,0,0,\dots,0,1).
\end{aligned}$$

The second and the third rows in this system vanish in the case $n = 1$. Then we arrive at the single nonlinear elliptic parametric mean-curvature equation

$$\operatorname{div}\frac{\nabla\zeta}{W} = 2H(X,N) \quad \text{in } \Omega.$$

The proof follows from our considerations from chapter 10.

Representation of quasilinear equations in divergence form due to Bers

Now let again $n = 1$. Non-parametric variational problems

$$\iint\limits_{\Omega} F(\zeta_x, \zeta_y)\,dxdy$$

lead to Euler-Lagrange equations in divergence form

$$\frac{d}{dx}F_p(\zeta_x,\zeta_y) + \frac{d}{dy}F_q(\zeta_x,\zeta_y) = 0 \quad \text{in } \Omega.$$

We now follow Bers [14] and consider quasilinear equations

$$A(p,q)r + 2B(p,q)s + C(p,q)t = 0 \quad \text{in } \Omega$$

with the additional settings

$$r := \zeta_{xx}, \quad s := \zeta_{xy}, \quad t := \zeta_{yy}.$$

Assume that the coefficents A, B and C as well as its first derivatives are Hölder continuous in a $[p,q]$-domain containing the origin $(0,0)$.

Now Bers proved that *under these conditions we can always rewrite the above quasiliner elliptic equation in divergence form*

$$\frac{d\lambda(p,q)}{dx} + \frac{d\mu(p,q)}{dy} = 0.$$

Necessary and sufficient for this is the existence of non-vanishing and positive function $\chi = \chi(p,q)$ with the properties

$$\lambda_p = \chi A, \quad \lambda_p + \mu_q = 2\chi B, \quad \mu_q = \chi C.$$

Then the complex-valued function $\kappa := \lambda + i\mu$ is a solution of the system

$$\lambda_p = \frac{A}{C}\,\mu_q\,, \quad \lambda_q = -\mu_p + \frac{2B}{C}\,\mu_q\,.$$

Proposition 11.4. *(Bers [14])*
Under the above assumptions there exists a continuously differentiable homeomorphism $\kappa = \lambda + i\mu$ solving the foregoing system such that

$$\lambda_p > 0,$$
$$\lambda = \mu = \lambda_p = 0, \quad \lambda_p = 1 \quad \textit{in } p = q = 0,$$
$$p\lambda + q\mu > 0 \quad \textit{for } p^2 + q^2 > 0,$$
$$\lambda_p \mu_q - \lambda_q \mu_p > 0.$$

On the size of quasilinear graphs

Consider now the divergence equation

$$\frac{d}{dx} F_p(x,y,\zeta,\zeta_x,\zeta_y) + \frac{d}{dy} F_q(x,y,\zeta,\zeta_x,\zeta_y) = F_z(x,y,\zeta,\zeta_x,\zeta_y) \tag{11.9}$$

on the closed disc

$$B_R := \left\{(x,y) \in \mathbb{R}^2 \, : \, x^2 + y^2 \leq R^2\right\}.$$

Theorem 11.4. *Let $\zeta \in C^2(\mathring{B}_R,\mathbb{R}) \cap C^1(B_R,\mathbb{R})$ be a solution of (11.9). Suppose that*

$$F_p(x,y,z,p,q)^2 + F_q(x,y,z,p,q)^2 \leq k_0^2\,,$$
$$F_z(x,y,z,p,q) \geq F_{min}$$
$$\textit{for all } (x,y,z,p,q) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

with real constants $k_0 \in [0,+\infty)$ and $F_{min} \in (0,+\infty)$. Then it holds

$$R \leq \frac{2k_0}{F_{min}}\,.$$

Proof. Partial integration of the divergence equation yields

$$\iint\limits_{B_R} F_z\,dxdy = \iint\limits_{B_R} \operatorname{div}(F_p,F_q)\,dxdy = \int\limits_{\partial B_R} (-F_q\,dx + F_p\,dy).$$

Now on the one hand we know

$$F_{min}\pi R^2 \leq \iint\limits_{B_R} F_z\,dxdy,$$

while on the other hand it holds

$$\int\limits_{\partial B_R} (-F_q\,dx + F_q\,dy) \le \int\limits_{\partial B_R} \sqrt{F_p^2 + F_q^2}\,\sqrt{dx^2 + dy^2} \le 2\pi k_0 R.$$

Comparing both inequalities proves the statement. □

Our method follows the lines of Heinz [79] where graphs with prescribed scalar mean curvature are considered.

Quasilinear elliptic equations and immersions of mean curvature type

Let $\zeta \in C^{3+\alpha}(\Omega, \mathbb{R})$ be a solution of the differential equation

$$A(x,y,\zeta,\nabla\zeta)\zeta_{xx} + 2B(x,y,\zeta,\nabla\zeta)\zeta_{xy} + C(x,y,\zeta,\nabla\zeta)\zeta_{yy} = D(x,y,\zeta,\nabla\zeta) \quad (11.10)$$

in Ω where w.l.o.g. we assume

$$\Lambda^2 := AC - B^2 = 1 + |\nabla\zeta|^2\,.$$

The unit normal vector of such a graph $X(x,y) = (x,y,\zeta(x,y))$ reads as

$$N(x,y) = \frac{1}{\sqrt{1 + |\nabla\zeta|^2}}\,(-\zeta_x, -\zeta_y, 1).$$

Following ideas of Sauvigny [141] we want to construct a weight matrix $\mathbf{W}(X,N)$ which transforms (11.10) int a system of mean curvature type

$$\nabla_{ds_W^2}(X,N) = -2H_W(X,N).$$

Construction of the weight matrix

We start with

$$\mathbf{M}(x,y) := \begin{pmatrix} \begin{pmatrix} 1+\zeta_x^2 & \zeta_x\zeta_y \\ \zeta_x\zeta_y & 1+\zeta_y^2 \end{pmatrix}^{-1} \circ \begin{pmatrix} X_x \\ X_y \end{pmatrix} \\ N(x,y) \end{pmatrix}$$

and define the weight matrix

$$\mathbf{W}(X,N) := \mathbf{M}(x,y)^T \circ \begin{pmatrix} C & -B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \mathbf{M}(x,y). \quad (11.11)$$

Let us call the matrix in the middle of the right hand side as $\mathbf{K} = \mathbf{K}(x,y)$.

Lemma 11.2. *It holds*

$$\det \mathbf{W}(X,N) = 1.$$

Proof. We compute

$$\begin{pmatrix} 1+\zeta_x^2 & \zeta_x\zeta_y \\ \zeta_x\zeta_y & 1+\zeta_y^2 \end{pmatrix}^{-1} \circ \begin{pmatrix} 1 & 0 & \zeta_x \\ 0 & 1 & \zeta_y \end{pmatrix} = \frac{1}{\Lambda^2} \begin{pmatrix} 1+\zeta_y^2 & -\zeta_x\zeta_y & \zeta_x \\ -\zeta_x\zeta_y & 1+\zeta_x^2 & \zeta_y \end{pmatrix},$$

and therefore

$$\mathbf{M}(x,y) = \frac{1}{\Lambda^2} \begin{pmatrix} 1+\zeta_y^2 & -\zeta_x\zeta_y & \zeta_x \\ -\zeta_x\zeta_y & 1+\zeta_x^2 & \zeta_y \\ -\Lambda\zeta_x & -\Lambda\zeta_y & \Lambda \end{pmatrix}. \tag{11.12}$$

Its determinant reads

$$\det \mathbf{M}(x,y) = \frac{1}{\Lambda^6}\Lambda(1+\zeta_x^2+\zeta_y^2)^2 = \frac{1}{\Lambda^5}\Lambda\cdot\Lambda^4 = \frac{1}{\Lambda}.$$

Considering

$$\det \mathbf{W}(X,N) = \det \mathbf{M}(x,y)\cdot\det \mathbf{K}(x,y)\cdot\det \mathbf{M}(x,y) = \Lambda^{-1}\Lambda^2\Lambda^{-1} = 1$$

the statement follows. □

Lemma 11.3. *It holds*

$$\mathbf{W}(X,N)\circ N = N.$$

Proof. Using (11.12) we compute

$$\frac{1}{\Lambda}\mathbf{M}(x,y)\circ(-\zeta_x,-\zeta_y,1) = (0,0,1)$$

as well as

$$\frac{1}{\Lambda}\mathbf{K}(x,y)\circ\mathbf{M}(x,y)\circ(-\zeta_x,-\zeta_y,1) = \mathbf{K}(x,y)\circ(0,0,1) = (0,0,1),$$

and the statement follows from (11.11). □

Lemma 11.4. *It holds*

$$\begin{pmatrix} C & -B \\ -B & A \end{pmatrix} = \partial X^T \circ \mathbf{W}(X,N)\circ\partial X. \tag{11.13}$$

Proof. Since

$$\mathbf{M}(x,y)\circ\partial X=\frac{1}{\Lambda^2}\begin{pmatrix}1+\zeta_y^2 & -\zeta_x\zeta_y & \zeta_x\\ -\zeta_x\zeta_y & 1+\zeta_x^2 & \zeta_y\\ -\Lambda\zeta_x & -\Lambda\zeta_y & \Lambda\end{pmatrix}\circ\begin{pmatrix}1 & 0\\ 0 & 1\\ \zeta_x & \zeta_y\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$$

we arrive at

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\circ\begin{pmatrix}C & -B & 0\\ -B & A & 0\\ 0 & 0 & 1\end{pmatrix}\circ\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}C & -B\\ -B & A\end{pmatrix}$$

proving the statement. □

Transformation into a weighted mean curvature system

We now define the weighted metrical coefficients

$$h_{11}(x,y):=C,\quad h_{12}(x,y):=-B,\quad h_{22}(x,y):=A \tag{11.14}$$

and calculate

$$h^{11}(x,y)=\frac{A}{\Lambda^2},\quad h^{12}(x,y)=\frac{B}{\Lambda^2},\quad h^{22}(x,y)=\frac{C}{\Lambda^2}.$$

Thus we can rewrite the differential equation in the form

$$h^{11}(X_{xx}\cdot N)+2h^{12}(X_{xy}\cdot N)+h^{22}(X_{yy}\cdot N)=\sum_{i,j=1}^{2}h^{ij}L_{ij}=\left\{1+|\nabla\zeta|^2\right\}^{-\frac{3}{2}}D.$$

For the weighted mean curvature we set

$$H_W(X,N):=\frac{1}{2}\left\{1+|\nabla\zeta|^2\right\}^{-\frac{3}{2}}D(x,y,\zeta,\nabla\zeta). \tag{11.15}$$

Theorem 11.5. *With the settings (11.11), (11.14) and (11.15), any solution* $\zeta\in C^{3+\alpha}(\Omega,\mathbb{R})$ *of the quasilinear elliptic equation (11.10) can be transformed into the weighted mean curvature system*

$$\nabla_{ds_W^2}(X,N)=-2H_W(X,N)\quad \textit{in } \Omega.$$

An analogous connection was already established in Sauvigny [141] for the homogeneous case of vanishing weighted mean curvature with $D\equiv 0$.

Ellipticity of the weight matrix

Taking account of (11.13), the ellipticity constant $\omega_0 \geq 0$ can be realized as follows (again we refer to Sauvigny [141]):

$$\frac{1}{1+\omega_0} \leq \frac{\xi \circ \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} \circ \xi}{\xi \circ \begin{pmatrix} 1+\zeta_x^2 & \zeta_x\zeta_y \\ \zeta_x\zeta_y & 1+\zeta_y^2 \end{pmatrix} \circ \xi} \leq 1+\omega_0 \tag{11.16}$$

for all $\xi \in \mathbb{R}^2 \setminus \{0\}$.

A growth estimate of the form (11.16) can already be found in works of Bernstein and Finn (see e.g. the analysis in Nitsche [126], §575 ff.). Namely with

$$\xi = (\xi_1, \xi_2), \quad p = \zeta_x \quad \text{and} \quad q = \zeta_y$$

we consider the two metrical elements

$$\begin{aligned} ds_1^2 &:= (1+p^2)\,d\xi_1^2 + 2pq\,d\xi_1 d\xi_2 + (1+q^2)\,d\xi_2^2\,, \\ ds_2^2 &:= C\,d\xi_1^2 - 2B\,d\xi_1 d\xi_2 + A\,d\xi_2^2\,. \end{aligned}$$

If (11.16) holds true, the elements ds_1^2 and ds_2^2 are said to be in *quasiconformal relation* which means that for their *dilatation* δ satisfies

$$\frac{1}{(1+\omega_0)^2} \leq \delta := \frac{\left(\frac{ds_2}{ds_1}\right)_{max}}{\left(\frac{ds_1}{ds_2}\right)_{min}} \leq (1+\omega_0)^2\,.$$

Following Finn [58] we speak of a *graph of minimal surface type* if $D \equiv 0$, or of *graphs of mean curvature type* in the non-homogeneous case.

Let us finally consider the example of the mean curvature equation

$$\begin{aligned} (1+\zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1+\zeta_x^2)\zeta_{yy} \\ = 2H(x,y,\zeta,\zeta_x,\zeta_y)(1+|\nabla\zeta|^2)^{\frac{3}{2}} \end{aligned}$$

to illustrate the calculations so far.

For the weight matrix from (11.11) we compute

$$\frac{1}{\Lambda^4}\begin{pmatrix}1+\zeta_y^2 & -\zeta_x\zeta_y & -\Lambda\,\zeta_x\\ -\zeta_x\zeta_y & 1+\zeta_x^2 & -\Lambda\,\zeta_y\\ \zeta_x & \zeta_y & \Lambda\end{pmatrix}\circ\begin{pmatrix}1+\zeta_x^2 & \zeta_x\zeta_y & 0\\ \zeta_x\zeta_y & 1+\zeta_y^2 & 0\\ 0 & 0 & 1\end{pmatrix}\circ\begin{pmatrix}1+\zeta_y^2 & -\zeta_x\zeta_y & \zeta_x\\ -\zeta_x\zeta_y & 1+\zeta_x^2 & \zeta_y\\ -\Lambda\,\zeta_x & -\Lambda\,\zeta_y & \Lambda\end{pmatrix}$$

$$=\frac{1}{\Lambda^4}\begin{pmatrix}1+\zeta_y^2 & -\zeta_x\zeta_y & -\Lambda\,\zeta_x\\ -\zeta_x\zeta_y & 1+\zeta_x^2 & -\Lambda\,\zeta_y\\ \zeta_x & \zeta_y & \Lambda\end{pmatrix}\circ\begin{pmatrix}\Lambda^4 & 0 & \Lambda^2\zeta_x\\ 0 & \Lambda^4 & \Lambda^2\zeta_y\\ -\Lambda\,\zeta_x & -\Lambda\,\zeta_y & \Lambda\end{pmatrix}$$

$$=\frac{1}{\Lambda^4}\begin{pmatrix}\Lambda^4 & 0 & 0\\ 0 & \Lambda^4 & 0\\ 0 & 0 & \Lambda^4\end{pmatrix},$$

i.e. $\mathbf{W}(X,N)\equiv\mathbb{E}^3$, and we may choose $\omega_0=0$.

11.6 Quasilinear elliptic systems

In this section we want to derive various elliptic systems for the surface vector X of a weighted conformally parametrized immersion of mean curvature type and for its spherical mapping N. Based on these identities we particularly elaborate the quadratic growth of the gradients in the next section which becomes important for the later curvature estimates.

An example: Minimal surfaces

For illustration we want to start with considering a minimal surface $X\colon B\to\mathbb{R}^3$ satisfying the conformality relation

$$X_u^2=W=X_v^2\,,\quad X_u\cdot X_v=0\quad\text{in } B.$$

Let $N\colon B\to\mathbb{R}^3$ denote the Gauss map of the minimal surface. From the invariant representation

$$\nabla_{ds^2}(X,N)=0\quad\text{in } B$$

w.r.t. the non-weighted metric $ds^2=g_{11}\,du^2+2g_{12}\,du dv+g_{22}\,dv^2$ we infer

$$\Delta X\cdot N=0\quad\text{in } B.$$

Furthermore, differentiating the conformality relations gives us

$$X_u \cdot X_{uu} = X_v \cdot X_{uv}\,, \quad X_u \cdot X_{uv} = X_v \cdot X_{vv}\,,$$

$$X_u \cdot X_{uv} + X_v \cdot X_{uu} = 0, \quad X_u \cdot X_{vv} + X_v \cdot X_{uv} = 0$$

from where we get

$$X_u \cdot X_{uu} = -X_u \cdot X_{vv}\,, \quad X_v \cdot X_{uu} = -X_v \cdot X_{vv}\,.$$

Thus there also hold

$$\Delta X \cdot X_u = 0, \quad \Delta X \cdot X_v = 0,$$

and we arrive at the *conformally minimal surface system*

$$\Delta X = 0 \quad \text{in } B.$$

In particular, *the coordinate functions* $x^i = x^i(u,v)$, $i = 1,2,3$, *represent harmonic functions.*

Weighted minimal surfaces

From (4.11) we already know the representation

$$\Delta_W X = \Delta X - (\Omega_{11}^1 + \Omega_{22}^1)X_u - (\Omega_{11}^2 + \Omega_{22}^2)X_v = 2H_W(X,N)WN$$

using weighted conformal parameters $(u,v) \in B$ setting

$$\Omega_{ij}^k = -\frac{1}{2}\sum_{\ell=1}^{2} h^{k\ell}(\omega_{\ell ij} + \omega_{j\ell i} - \omega_{ij\ell}), \quad \omega_{ij\ell} = X_{u^i} \circ \mathbf{W}(X,N)_{u^\ell} \circ X_{u^j}$$

with the coefficients

$$h_{ij} = X_{u^i} \circ \mathbf{W}(X,N) \circ X_{u^j}$$

of the weighted first fundamental form. In the non-weighted case $\mathbf{W}(X,N) = \mathbb{E}^3$ this system reduces to the classical minimal surface system $\Delta X = 0$.

Elliptic systems for the spherical mapping

The differential system for the surface vector X from the last example contains the spherical mapping N and its first derivatives. Thus to arrive at a complete system of equations we must establish suitable differential equations for the vector N.

Theorem 11.6. *Let the weighted conformally parametrized immersion* $X\colon B \to \mathbb{R}^3$ *with prescribed weighted mean curvature* $H_W(X,Z)$ *be given.*

Then its spherical mapping $N\colon B \to \mathbb{R}^3$ satisfies

$$\begin{aligned}\Delta N = {} & 2\,(N_u \times N_v) + \mathbf{W}(X,N)_u \circ (N \times N_v) + \mathbf{W}(X,N)_v \circ (N_u \times N) \\ & -2\left\{H_{W,X}(X,N)\cdot X_u + H_{W,Z}(X,N)\cdot N_u\right\}\mathbf{W}(X,N)\circ X_u \\ & -2\left\{H_{W,X}(X,N)\cdot X_v + H_{W,Z}(X,N)\cdot N_v\right\}\mathbf{W}(X,N)\circ X_v \\ & -2H_W(X,N)\left\{\mathbf{W}(X,N)_u \circ X_u + \mathbf{W}(X,N)_v \circ X_v\right\} \\ & -2H_W(X,N)\,\mathbf{W}(X,N)\circ \Delta X\,. \end{aligned} \tag{11.17}$$

For the proof we need the following calculus rule (see Sauvigny [141]).

Lemma 11.5. *Using weigthed conformal parameters $(u,v) \in B$ it holds*

$$\mathbf{W}(X,N)\circ X_u = (X_v \times N)\,, \quad \mathbf{W}(X,N)\circ X_v = (N \times X_u). \tag{11.18}$$

Proof of the Lemma. The vector triple

$$\left\{\frac{1}{\sqrt{W}}\,\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u, \frac{1}{\sqrt{W}}\,\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v, N\right\}$$

forms an orthonormal moving frame of the surface. It particularly holds

$$\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v = N \times \mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u\,.$$

Using the identity $\mathbf{W}(X,N)\circ N = N$ and the rule from Lemma 5.3 from chapter 5 we compute

$$\begin{aligned}\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_v &= \left\{\mathbf{W}(X,N)^{\frac{1}{2}}\circ N\right\} \times \left\{\mathbf{W}(X,N)^{\frac{1}{2}}\circ X_u\right\} \\ &= \mathbf{W}(X,N)^{-\frac{1}{2}}\circ (N \times X_u)\end{aligned}$$

proving the second identity in (11.18). Furthermore we have

$$\begin{aligned}X_u &= (N \times X_u) \times N = \left\{\mathbf{W}(X,N)\circ X_v\right\} \times \left\{\mathbf{W}(X,N)\circ N\right\} \\ &= \mathbf{W}(X,N)^{-1}\circ (X_v \times N)\,,\end{aligned}$$

and that was stated. □

Now we come to the

Proof of the Theorem. We write the Weingarten equations from section 4.3 using weighted conformal parameters $(u,v) \in B$ in the form

$$N_u = -\frac{L_{11}}{W}\,\mathbf{W}(X,N)\circ X_u - \frac{L_{12}}{W}\,\mathbf{W}(X,N)\circ X_v\,,$$
$$N_v = -\frac{L_{21}}{W}\,\mathbf{W}(X,N)\circ X_u - \frac{L_{22}}{W}\,\mathbf{W}(X,N)\circ X_v\,.$$

Together with (11.18) we calculate

$$\begin{aligned} N\times N_u &= -\frac{L_{11}}{W}\,N\times \mathbf{W}(X,N)\circ X_u - \frac{L_{12}}{W}\,N\times\mathbf{W}(X,N)\circ X_v \\ &= -\frac{L_{11}}{W}\,N\times(X_v\times N) - \frac{L_{12}}{W}\,N\times(N\times X_u) \\ &= -\frac{L_{11}}{W}\,X_v + \frac{L_{12}}{W}\,X_u \end{aligned} \tag{11.19}$$

and analogously

$$N\times N_v = -\frac{L_{12}}{W}\,X_v + \frac{L_{22}}{W}\,X_u\,. \tag{11.20}$$

We conclude

$$\begin{aligned} N\times N_u &= \frac{L_{12}}{W}\,X_u + \frac{L_{22}}{W}\,X_v - \frac{L_{11}+L_{22}}{W}\,X_v \\ &= -\mathbf{W}(X,N)^{-1}\circ N_v - 2H_W(X,N)X_v \end{aligned}$$

and

$$\begin{aligned} N\times N_v &= -\frac{L_{12}}{W}\,X_v - \frac{L_{11}}{W}\,X_u + \frac{L_{11}+L_{22}}{W}\,X_u \\ &= \mathbf{W}(X,N)^{-1}\circ N_u + 2H_W(X,N)X_u\,. \end{aligned}$$

Rearranging yields

$$\begin{aligned} N_u &= \mathbf{W}(X,N)\circ(N\times N_v) - 2H_W(X,N)\,\mathbf{W}(X,N)\circ X_u\,, \\ N_v &= -\mathbf{W}(X,N)\circ(N\times N_u) - 2H_W(X,N)\,\mathbf{W}(X,N)\circ X_v\,. \end{aligned}$$

Now differentiate these relations to get

$$\begin{aligned} N_{uu} = {} & \mathbf{W}(X,N)_u\circ(N\times N_v) + \mathbf{W}(X,N)\circ(N_u\times N_v) \\ & + \mathbf{W}(X,N)\circ(N\times N_{uv}) - 2H_W(X,N)\,\mathbf{W}(X,N)_u\circ X_u \\ & - 2H_W(X,N)\,\mathbf{W}(X,N)\circ X_{uu} - 2H_W(X,N)_u\,\mathbf{W}(X,N)\circ X_u \end{aligned}$$

as well as

$$\begin{aligned} N_{vv} = & -\mathbf{W}(X,N)_v \circ (N \times N_u) - \mathbf{W}(X,N) \circ (N_v \times N_u) \\ & -\mathbf{W}(X,N) \circ (N \times N_{uv}) - 2H_W(X,N)\,\mathbf{W}(X,N)_v \circ X_v \\ & -2H_W(X,N)\,\mathbf{W}(X,N) \circ X_{vv} - 2H_W(X,N)_v\,\mathbf{W}(X,N) \circ X_v\,. \end{aligned}$$

Summing up both identities proves the statement. □

We want to evaluate the normal part of the term

$$\mathbf{W}(X,N)_u \circ (N \times N_v) + \mathbf{W}(X,N)_v \circ (N_u \times N).$$

Let as usual $[X,Y,Z] := X \cdot (Y \times Z)$. Then after differentiation of $N = \mathbf{W}(X,N) \circ N$ we compute

$$\begin{aligned} & N \circ \mathbf{W}(X,N)_u \circ (N \times N_v) + N \circ \mathbf{W}(X,N)_v \circ (N_u \times N) \\ & = (N \times N_v) \circ \mathbf{W}(X,N)_u \circ N + (N_u \times N) \circ \mathbf{W}(X,N)_v \circ N \\ & = (N \times N_v) \cdot N_u - (N \times N_v) \circ \mathbf{W}(X,N) \circ N_u \\ & \quad + (N_u \times N) \cdot N_v - (N_u \times N) \circ \mathbf{W}(X,N) \circ N_v \\ & = -2\,[N,N_u,N_v] \\ & \quad - \big[\mathbf{W}(X,N) \circ N, N_u, N_v\big] + \big[\mathbf{W}(X,N) \circ N, N_u, N_v\big] \\ & \quad + \big[N, \mathbf{W}(X,N) \circ N_u, N_v\big] + \big[N, N_u, \mathbf{W}(X,N) \circ N_v\big] \\ & = \Big\{\operatorname{trace}\mathbf{W}(X,N) - 3\Big\}\,[N,N_u,N_v] \end{aligned}$$

where $[N,N_u,N_v] = KW$. *Thus this part vanishes identically in the non-weighted case* $\mathbf{W}(X,Z) \equiv \mathbb{E}^3$.

An alternative elliptic system for the spherical mapping

The identites from (11.18) enable us to derive an elliptic system for X : Differentiate

$$X_u = \mathbf{W}(X,N)^{-1} \circ (X_v \times N)\,, \quad X_v = \mathbf{W}(X,N)^{-1} \circ (N \times X_u)$$

to arrive at

$$\begin{aligned} X_{uu} &= \Big\{\mathbf{W}(X,N)^{-1}\Big\}_u \circ (X_v \times N) + \mathbf{W}(X,N)^{-1} \circ (X_{uv} \times N + X_v \times N_u)\,, \\ X_{vv} &= \Big\{\mathbf{W}(X,N)^{-1}\Big\}_v \circ (N \times X_u) + \mathbf{W}(X,N)^{-1} \circ (N_v \times X_u + N \times X_{uv})\,. \end{aligned}$$

and summing up these identities shows that

$$\begin{aligned}\Delta X &= \left\{\mathbf{W}(X,N)^{-1}\right\}_u \circ (X_v \times N) + \left\{\mathbf{W}(X,N)^{-1}\right\}_v \circ (N \times X_u)\\ &\quad + N_v \times X_u + X_v \times N_u\,.\end{aligned}$$

The third term on the right hand side does not involves any tangential parts. Thus from (11.19) and (11.20) we infer

$$\begin{aligned}N \cdot (X_u \times N_v + N_u \times X_v) &= -X_u \cdot (N \times N_v) - X_v \cdot (N_u \times N)\\ &= -\frac{L_{22}g_{22} - 2L_{12}g_{12} + L_{11}g_{11}}{W} = -2HW\end{aligned}$$

with the non-weighted mean curvature $H = H(u,v)$ – *but using weighted conformal parameters.* Together with (11.18) we conclude

$$\begin{aligned}\Delta X &= \left\{\mathbf{W}(X,N)^{-1}\right\}_u \circ \mathbf{W}(X,N) \circ X_u + \left\{\mathbf{W}(X,N)^{-1}\right\}_v \circ \mathbf{W}(X,N) \circ X_v + 2HWN\\ &= -\mathbf{W}(X,N)^{-1} \circ \mathbf{W}(X,N)_u \circ X_u - \mathbf{W}(X,N)^{-1} \circ \mathbf{W}(X,N)_v \circ X_v + 2HWN\end{aligned}$$

or after rearranging

$$\mathbf{W}(X,N) \circ \Delta X = -\mathbf{W}(X,N)_u \circ X_u - \mathbf{W}(X,N)_v \circ X_v + 2HWN.$$

This could eventually serve as an alternative system for the surface vector X. Furthermore, in the non-weighted case $\mathbf{W}(X,Z) \equiv \mathbb{E}^3$ we immediately read the mean curvature system

$$\Delta X = 2HWN$$

in conformal parameters.

Inserting next this representation of $\mathbf{W}(X,N) \circ \Delta X$ into (11.17), taking account of $N_u \times N_v = KWN$, proves the

Corollary 11.2. *Let the weighted conformally parametrized immersion $X : B \to \mathbb{R}^3$ with prescribed weighted mean curvature $H_W(X,Z)$ be given. Then its spherical mapping satisfies*

$$\begin{aligned}\Delta N &= -2\left\{2HH_W(X,N) - K\right\}WN\\ &\quad + \mathbf{W}(X,N)_u \circ (N \times N_v) + \mathbf{W}(X,N)_v \circ (N_u \times N)\\ &\quad - 2\left\{H_{W,X}(X,N) \cdot X_u + H_{W,Z}(X,N) \cdot N_u\right\}\mathbf{W}(X,N) \circ X_u\\ &\quad - 2\left\{H_{W,X}(X,N) \cdot X_v + H_{W,Z}(X,N) \cdot N_v\right\}\mathbf{W}(X,N) \circ X_v\end{aligned} \tag{11.21}$$

with the non-weighted mean curvature H.

Examples

Let us exemplarily consider the case $\mathbf{W}(X,Z)=\mathbb{E}^3$. Using conformal parameters we infer

$$\begin{aligned}\Delta N=&-2\left\{2H(X,N)^2-K\right\}WN\\&-2\left\{H_X(X,N)\cdot X_u+H_Z(X,N)\cdot N_u\right\}X_u\\&-2\left\{H_X(X,N)\cdot X_v+H_Z(X,N)\cdot N_v\right\}X_v\,.\end{aligned}$$

Note the coupling with the gradient of the surface vector.

If $X\colon B\to\mathbb{R}^3$ represents a minimal surface then it holds

$$\Delta N=2KWN=2(N_u\times N_v).$$

We want to express ΔN in terms of the moving frame $\{N,N_u,N_v\}$. Using conformal parameters we have

$$\begin{aligned}H_X&=\left(H_X\cdot\frac{X_u}{|X_u|}\right)\frac{X_u}{|X_u|}+\left(H_X\cdot\frac{X_v}{|X_v|}\right)\frac{X_v}{|X_v|}+\big(H_X\cdot N\big)N\\&=\frac{1}{W}\left\{\big(H_X\cdot X_u\big)X_u+\big(H_X\cdot X_v\big)X_v\right\}+\big(H_X\cdot N\big)N\end{aligned}$$

w.r.t. the basis $\{X_u,X_v,N\}$ such that it follows

$$\begin{aligned}\Delta N=&\,2\left\{K-2H(X,N)^2+H_X(X,N)\cdot N\right\}WN-2H_X(X,N)W\\&-2\left\{H_Z(X,N)\cdot N_u\right\}X_u-2\left\{H_Z(X,N)\cdot N_v\right\}X_v\,.\end{aligned}\tag{11.22}$$

Now compare the terms

$$\begin{aligned}&(H_Z\cdot N_u)X_u+(H_Z\cdot N_v)X_v\\&\quad=-\frac{L_{11}}{W}(H_Z\cdot X_u)X_u-\frac{L_{12}}{W}(H_Z\cdot X_v)X_u-\frac{L_{12}}{W}(H_Z\cdot X_u)X_v-\frac{L_{22}}{W}(H_Z\cdot X_v)X_v\,,\\&(H_Z\cdot X_u)N_u+(H_Z\cdot X_v)N_v\\&\quad=-\frac{L_{11}}{W}(H_Z\cdot X_u)X_u-\frac{L_{12}}{W}(H_Z\cdot X_u)X_v-\frac{L_{12}}{W}(H_Z\cdot X_v)X_u-\frac{L_{22}}{W}(H_Z\cdot X_v)X_v\end{aligned}$$

showing the identity

$$(H_Z\cdot N_u)X_u+(H_Z\cdot N_v)X_v=(H_Z\cdot X_u)N_u+(H_Z\cdot X_v)N_v\,.$$

Thus we can rewrite (11.22) into the form

$$\begin{aligned}\Delta N = 2\left\{K - 2H(X,N)^2 + H_X(X,N)\cdot N\right\}WN - 2H_X(X,N)W \\ -2\left\{H_Z(X,N)\cdot X_u\right\}N_u - 2\left\{H_Z(X,N)\cdot X_v\right\}N_v\,.\end{aligned}$$

Finally we want to specify (11.21) for weighted minimal surfaces satisfying $H_W(X,Z)\equiv 0$: We immediately infer

$$\Delta N = 2KWN + \mathbf{W}(X,N)_u \circ (N\times N_v) + \mathbf{W}(X,N)_v \circ (N_u \times N).$$

Note again the coupling with the gradient of the surface vector. This coupling vanishes in the special case $\mathbf{W} = \mathbf{W}(N)$.

11.7 Quadratic growth in the gradient

In this section we elaborate the quadratic growth in the gradient for the elliptic systems from above.

Estimate of the Ω_{ij}^k

Beside the constant $\omega_0 \in [0,+\infty)$ we also need quantities controlling the derivatives of the weight matrix.

Definition 11.4. Let $\omega_1,\omega_2 \in [0,+\infty)$ be real constants with the properties

$$\sqrt{\sum_{i,j,k=1}^{3} w_{ij,x^k}(X,Z)^2} \le \omega_1$$

as well as

$$\sqrt{\sum_{i,j,k=1}^{3} w_{ij,z^k}(X,Z)^2} \le \omega_2$$

with the components $w_{ij}(X,Z)$ of the weight matrix $\mathbf{W}(X,Z)$.

The coupling of the surface vector X and its spherical mapping N through the weight matrix $\mathbf{W}(X,N)$ causes a coupling of the associated elliptic systems. Thus our next definition.

Definition 11.5. With a real number $r>0$ we set

$$\mathfrak{X}(u,v) := \left(\frac{1}{r}X(u,v), N(u,v)\right), \quad (u,v)\in B.$$

In terms of this system we want to establish estimates for the weight matrix.

Lemma 11.6. *Let some vector* $\xi \in \mathbb{R}^3$ *be given. Then it holds*

$$|\mathbf{W}(X,Z)_{u^m} \circ \xi| \le |\xi| \sqrt{2(r\omega_1)^2 + 2\omega_2^2}\, |\mathfrak{X}_{u^m}|, \quad m = 1,2. \tag{11.23}$$

Proof. For $\xi = (\xi^1, \xi^2, \xi^3)$ we compute

$$\begin{aligned}
\big|\mathbf{W}(X,Z)_{u^m} \circ \xi\big|^2 &= \sum_{i,j,k=1}^{3} \Big\{ w_{ij,x^k} x_{u^m}^k \xi^j + w_{ij,z^k} z_{u^m}^k \xi^j \Big\}^2 \\
&\le 2 \sum_{i,j,k=1}^{3} \Big\{ \Big(w_{ij,x^k} x_{u^m}^k \xi^j \Big)^2 + \Big(w_{ij,z^k} z_{u^m}^k \xi^j \Big)^2 \Big\} \qquad (11.24) \\
&\le 2|\xi|^2 \sum_{i,j,k=1}^{3} \Big\{ \Big(w_{ij,x^k} x_{u^m}^k \Big)^2 + \Big(w_{ij,z^k} z_{u^m}^k \Big)^2 \Big\}
\end{aligned}$$

due to the Hölder inequality, for example applied to

$$\left(\sum_{j=1}^{3} w_{ij,x^k} x_{u^m}^k \xi^j \right)^2 \le \sum_{j=1}^{3} \Big(w_{ij,x^k} x_{u^m}^k \Big)^2 \cdot \sum_{j=1}^{3} (\xi^j)^2 = |\xi|^2 \sum_{j=1}^{3} \Big(w_{ij,x^k} x_{u^m}^k \Big)^2 .$$

In this way we arrive at

$$\begin{aligned}
\big|\mathbf{W}(X,Z)_{u^m} \circ \xi\big|^2 &\le 2|\xi|^2 |X_{u^m}|^2 \sum_{i,j,k=1}^{3} (w_{ij,x^k})^2 + 2|\xi|^2 |Z_{u^m}|^2 \sum_{i,j,k=1}^{3} (w_{ij,z^k})^2 \\
&\le 2\omega_1^2 |\xi|^2 |X_{u^m}|^2 + 2\omega_2^2 |\xi|^2 |Z_{u^m}|^2 \\
&\le 2|\xi|^2 \Big\{ (r\omega_1)^2 + \omega_2^2 \Big\} \left\{ \left| \frac{1}{r} X_{u^m} \right|^2 + |Z_{u^m}|^2 \right\} \\
&\le 2|\xi|^2 \Big\{ (r\omega_1)^2 + \omega_2^2 \Big\} |\mathfrak{X}_{u^m}|^2
\end{aligned} \tag{11.25}$$

proving the statement. □

Lemma 11.7. *Using weighted conformal parameters* $(u,v) \in B$ *there hold*

$$|\Omega_{11}^1| \le \frac{(1+\omega_0)\sqrt{(r\omega_1)^2 + \omega_2^2}}{\sqrt{2}} |\mathfrak{X}_u|, \quad |\Omega_{22}^2| \le \frac{(1+\omega_0)\sqrt{(r\omega_1)^2 + \omega_2^2}}{\sqrt{2}} |\mathfrak{X}_v|$$

as well as

$$|\Omega^2_{11}| \le \frac{(1+\omega_0)\sqrt{(r\omega_1)^2+\omega_2^2}}{\sqrt{2}}\,(2|\mathfrak{X}_u|+|\mathfrak{X}_v|),$$

$$|\Omega^1_{22}| \le \frac{(1+\omega_0)\sqrt{(r\omega_1)^2+\omega_2^2}}{\sqrt{2}}\,(|\mathfrak{X}_u|+2|\mathfrak{X}_v|).$$

Proof. We estimate as follows

$$\begin{aligned}|\Omega^1_{11}| &\le \frac{1}{2W}\,|X_u\circ \mathbf{W}(X,N)_u\circ X_u| = \frac{|X_u|^2}{2W}\left|\frac{X_u}{|X_u|}\circ \mathbf{W}(X,N)_u\circ \frac{X_u}{|X_u|}\right| \\ &\le \frac{|X_u|^2}{2W}\left|\mathbf{W}(X,N)_u\circ \frac{X_u}{|X_u|}\right|,\end{aligned}$$

and together with (11.23) we infer

$$|\Omega^1_{11}| \le \frac{\sqrt{(r\omega_1)^2+\omega_2^2}\,|X_u|^2}{\sqrt{2}W}\,|\mathfrak{X}_u| \le \frac{(1+\omega_0)\sqrt{(r\omega_1)^2+\omega_2^2}}{\sqrt{2}}\,|\mathfrak{X}_u|.$$

Analogously we have

$$\Omega^2_{11} = -\frac{1}{W}\,X_u\circ \mathbf{W}(X,N)_u\circ X_v + \frac{1}{2W}\,X_u\circ \mathbf{W}(X,N)_v\circ X_u$$

and compute

$$\begin{aligned}|\Omega^2_{11}| &\le \frac{|X_u||X_v|}{W}\left|\frac{X_u}{|X_u|}\circ \mathbf{W}(X,N)_u\circ \frac{X_v}{|X_v|}\right| + \frac{|X_u|^2}{2W}\left|\frac{X_u}{|X_u|}\circ \mathbf{W}(X,N)_v\circ \frac{X_u}{|X_u|}\right| \\ &\le \frac{|X_u|^2+|X_v|^2}{2W}\,\sqrt{2(r\omega_1)^2+2\omega_2^2}\,|\mathfrak{X}_u| + \frac{|X_u|^2}{2W}\,\sqrt{2(r\omega_1)^2+2\omega_2^2}\,|\mathfrak{X}_v| \\ &\le (1+\omega_0)\,\frac{X_u\circ \mathbf{W}(X,N)\circ X_u + X_v\circ \mathbf{W}(X,N)\circ X_v}{2W}\,\sqrt{2(r\omega_1)^2+2\omega_2^2}\,|\mathfrak{X}_u| \\ &\quad + (1+\omega_0)\,\frac{X_u\circ \mathbf{W}(X,N)\circ X_u}{2W}\,\sqrt{2(r\omega_1)^2+2\omega_2^2}\,|\mathfrak{X}_v|,\end{aligned}$$

or summarized

$$|\Omega^2_{11}| \le \frac{(1+\omega_0)\sqrt{(r\omega_1)^2+\omega_2^2}}{\sqrt{2}}\,\big(2|\mathfrak{X}_u|+|\mathfrak{X}_v|\big).$$

The remaining inequalities are proved in the same way. □

Estimate of ΔX

From the weighted mean curvature system we immediately infer

$$|\Delta X| \le (|\Omega_{11}^1| + |\Omega_{22}^1|)|X_u| + (|\Omega_{11}^2| + |\Omega_{22}^2|)|X_v| + h_0|\nabla X|^2$$

with the constant

$$h_0 := \sup_{\substack{X\in\mathbb{R}^3 \\ Z\in\mathbb{R}^3\setminus\{0\}}} |H_W(X,Z)|.$$

Here we note

$$W = \sqrt{h_{11}h_{22} - h_{12}^2} = \sqrt{g_{11}g_{22} - g_{12}^2} \le \sqrt{g_{11}g_{22}} \le \frac{1}{2}\,|\nabla X|^2\,.$$

Furthermore we have

$$\begin{aligned} |\Delta(r^{-1}X)| &\le (1+\omega_0)\sqrt{2(r\omega_1)^2 + 2\omega_2^2}\,(|\mathfrak{X}_u| + |\mathfrak{X}_v|)(|(r^{-1}X)_u| + |(r^{-1}X)_v|) \\ &\qquad + (h_0 r)\,|\nabla(r^{-1}X)|^2 \\ &\le (1+\omega_0)\sqrt{2(r\omega_1)^2 + 2\omega_2^2}\,(|\mathfrak{X}_u| + |\mathfrak{X}_v|)^2 + r(h_0 r)\,|\nabla\mathfrak{X}|^2\,. \end{aligned}$$

This proves our next

Theorem 11.7. *The weighted conformally parametrized immersion $X\colon B \to \mathbb{R}^3$ satisfies*

$$|\Delta(r^{-1}X)| \le \Lambda_X|\nabla\mathfrak{X}|^2 \quad \textit{in } B \tag{11.26}$$

with the constant

$$\Lambda_X := \left\{2(1+\omega_0)\sqrt{2(r\omega_1)^2 + 2\omega_2^2} + h_0 r\right\}. \tag{11.27}$$

Estimates of ΔN

First, system (11.17) yields

$$\begin{aligned} |\Delta N| &\le |\nabla N|^2 + \sqrt{2(r\omega_1)^2 + 2\omega_2^2}\,(|\mathfrak{X}_u||N\times N_v| + |\mathfrak{X}_v||N_u\times N|) \\ &\quad + 2(1+\omega_0)\left\{|H_{W,X}(X,N)||X_u| + |H_{W,Z}(X,N)||N_u|\right\}|X_u| \\ &\quad + 2(1+\omega_0)\left\{|H_{W,X}(X,N)||X_v| + |H_{W,Z}(X,N)||N_v|\right\}|X_v| \\ &\quad + 2h_0\sqrt{2(r\omega_1)^2 + 2\omega_2^2}\,(|\mathfrak{X}_u||X_u| + |\mathfrak{X}_v||X_v|) \\ &\quad + 2h_0(1+\omega_0)\,|\Delta X|. \end{aligned} \tag{11.28}$$

Next beside h_0 we also introduce real constants

$$h_1 := \sup_{\substack{X\in\mathbb{R}^3\\ Z\in\mathbb{R}^3\setminus\{0\}}} |H_{W,X}(X,Z)|, \quad h_2 := \sup_{\substack{X\in\mathbb{R}^3\\ Z\in\mathbb{R}^3\setminus\{0\}}} |H_{W,Z}(X,Z)|.$$

Together with (11.26) we then conclude

$$\begin{aligned}
|\Delta N| \le\; & |\nabla\mathfrak{X}|^2 + 2\sqrt{2(r\omega_1)^2+2\omega_2^2}\,|\mathfrak{X}_u||\mathfrak{X}_v| \\
& + 2(1+\omega_0)\Big\{(h_1r^2)|\mathfrak{X}_u|^2 + (h_2r)|\mathfrak{X}_u|^2\Big\} \\
& + 2(1+\omega_0)\Big\{(h_1r^2)|\mathfrak{X}_v|^2 + (h_2r)|\mathfrak{X}_v|^2\Big\} \\
& + 2(h_0r)\sqrt{2(r\omega_1)^2+2\omega_2^2}\,(|\mathfrak{X}_u|^2+|\mathfrak{X}_v|^2) \\
& + 2(h_0r)(1+\omega_0)\Big\{2(1+\omega_0)\sqrt{2(r\omega_1)^2+2\omega_2^2}+h_0r\Big\}\,|\nabla\mathfrak{X}|^2\,.
\end{aligned}$$

Theorem 11.8. *The spherical mapping N of a weighted conformally parametrized immersion $X\colon B\to\mathbb{R}^3$ satisfies*

$$|\Delta N| \le \Lambda_N|\nabla\mathfrak{X}|^2 \quad \textit{in } B \tag{11.29}$$

with the constant

$$\begin{aligned}
\Lambda_N := & 1+(1+2h_0r)\sqrt{2(r\omega_1)^2+2\omega_2^2}+2(1+\omega_0)\Big\{(h_1r^2)+(h_2r)\Big\} \\
& +2(h_0r)(1+\omega_0)\Big\{2(1+\omega_0)\sqrt{2(r\omega_1)^2+2\omega_2^2}+h_0r\Big\}.
\end{aligned} \tag{11.30}$$

Estimate of $\Delta\mathfrak{X}$

The estimates (11.26) and (11.29) immediately imply

Theorem 11.9. *Let the weighted conformally parametrized immersion $X\colon B\to\mathbb{R}^3$ be given. Then it holds*

$$|\Delta\mathfrak{X}| \le \Lambda_{\mathfrak{X}}|\nabla\mathfrak{X}|^2 =: \sqrt{2}\,(\Lambda_X+\Lambda_N)|\nabla\mathfrak{X}|^2 \quad \textit{in } B \tag{11.31}$$

with the constants Λ_X and Λ_N from (11.27), (11.30).

Eventually these estimates are not strong enough. We should rather consider special classes of surfaces.

Example: Minimal surfaces

From $\Delta X = 0$ and $\Delta N = 2(N_u \times N_v)$ we immediately obtain

$$|\Delta X| = 0, \quad |\Delta N| \le |\nabla N|^2 \quad \text{in } B$$

for a conformally parametrized minimal surface. In this case we have $\Lambda_X = 0$ and $\Lambda_N = 1$, but we replace (11.29) by (11.28) with $\omega_1, \omega_2 = 0$ and $H_W \equiv 0$.

Example: Surfaces with prescribed mean curvature

In this case there hold

$$\begin{aligned} &|\Delta(r^{-1}X)| \le (h_0 r)|\nabla(r^{-1}X)|^2\,, \\ &|\Delta N| \le (1+h_2 r)|\nabla N|^2 + \Big\{2(h_0^2+h_1)r^2 + h_2 r\Big\}|\nabla(r^{-1}X)|^2\,. \end{aligned}$$

In particular, for proving the second inequality we use (11.28) as follows

$$\begin{aligned} |\Delta N| &\le |\nabla N|^2 + 2h_0^2|\nabla X|^2 + 2h_1(|X_u|^2+|X_v|^2) + 2h_2(|X_u||N_u|+|X_v||N_v|) \\ &= |\nabla N|^2 + 2(h_0 r)^2|\nabla(r^{-1}X)|^2 + 2(h_1 r^2)|\nabla(r^{-1}X)|^2 \\ &\quad + 2(h_2 r)(|r^{-1}X_u||N_u| + |r^{-1}X_v||N_v|). \end{aligned}$$

We set $\Lambda_X = h_0 r$ and $\Lambda_N = 1 + 2(h_0 r)^2 + 2(h_1 r^2) + 2(h_2 r)$.

Special weighted minimal surfaces

Consider immersions with vanishing weighted mean curvature $H_W(X,Z) \equiv 0$. The estimates (11.26) and (11.29) imply

$$\begin{aligned} &|\Delta(r^{-1}X)| \le 2(1+\omega_0)\sqrt{2(r\omega_1)^2 + 2\omega_2^2}\;|\nabla\mathfrak{X}|^2\,, \\ &|\Delta N| \le \left\{1 + \sqrt{2(r\omega_1)^2 + 2\omega_2^2}\right\}|\nabla\mathfrak{X}|^2\,. \end{aligned}$$

Let us now assume that the weight matrix has the special structure

$$\mathbf{W} = \mathbf{W}(Z).$$

Examples are F-minimal surfaces which are critical for the anisotropic parametric functional $\mathscr{F}[X]$.

For arbitrary $\xi \in \mathbb{R}^3$ we replace the estimates (11.24) to (11.25) by

$$\big|\mathbf{W}(Z)_{u^m} \circ \xi\big|^2 = \sum_{i,j,k=1}^{3} \Big\{ w_{ij,z^k} z^k_{u^m} \xi^j \Big\}^2 \le |\xi|^2 \sum_{i,j,k=1}^{3} \Big\{ w_{ij,z^k} z^k_{u^m} \Big\}^2$$
$$\le |\xi|^2 |Z_{u^m}|^2 \sum_{i,j,k=1}^{3} w^2_{ij,z^k} \le \omega_2^2 |\xi|^2 |Z_{u^m}|^2 .$$

Furthermore we have

$$|\Omega^1_{11}| \le \frac{(1+\omega_0)\omega_2}{2} |N_u|, \quad |\Omega^2_{22}| \le \frac{(1+\omega_0)\omega_2}{2} |N_v|,$$
$$|\Omega^2_{11}| \le (1+\omega_0)\omega_2 |N_u| + \frac{(1+\omega_0)\omega_2}{2} |N_v|, \tag{11.32}$$
$$|\Omega^1_{22}| \le \frac{(1+\omega_0)\omega_2}{2} |N_u| + (1+\omega_0)\omega_2 |N_v|.$$

We will prove that in this special case $\mathbf{W} = \mathbf{W}(Z)$ the weighted minimal surface satisfies

$$|\Delta(r^{-1}X)| \le 2(1+\omega_0)\omega_2 |\nabla(r^{-1}X)||\nabla N|,$$
$$|\Delta N| \le (1+\omega_2)|\nabla N|^2 \tag{11.33}$$

using weighted conformal parameters. Namely, first we have

$$|\Delta X| \le (|\Omega^1_{11}| + |\Omega^1_{22}|)|X_u| + (|\Omega^2_{11}| + |\Omega^2_{22}|)|X_v|$$
$$\le (1+\omega_0)\omega_2 (|N_u| + |N_v|)(|X_u| + |X_v|)$$

showing the first inequality. Next we compute

$$\Delta N = 2(N_u \times N_v) + \mathbf{W}(N)_u \circ (N \times N_v) + \mathbf{W}(N)_v \circ (N_u \times N)$$

which leads us to

$$|\Delta N| \le 2|N_u||N_v| + \omega_2 |N_u||N_v| + \omega_2 |N_v||N_u| \le (1+\omega_2)|\nabla N|^2 ,$$

and this proves (11.33). *Note that X and N in (11.33) are not coupled.*

Example: Crystalline variational problems

We want to consider critical points $X\colon B \to \mathbb{R}^3$ of the parametric functional

$$\iint_B \Big\{ F(X_u \times X_v) + \frac{2\gamma_0}{3} X \cdot (X_u \times X_v) \Big\}\, dudv.$$

Note that the weight matrix has the structure $\mathbf{W} = \mathbf{W}(Z)$, thus $H_W = H_W(Z)$ for the weighted mean curvature. Weighted conformally parameterized critical points therefore fulfill

$$\begin{aligned}\Delta X &= (\Omega_{11}^1 + \Omega_{22}^1)X_u + (\Omega_{11}^2 + \Omega_{22}^2)X_v + 2H_W(N)WN, \\ \Delta N &= 2\,(N_u \times N_v) + \mathbf{W}(N)_u \circ (N \times N_v) + \mathbf{W}(N)_v \circ (N_u \times N) \\ &\quad - 2\left\{H_{W,Z}(N)\cdot N_u\right\}\mathbf{W}(N)\circ X_u - 2\left\{H_{W,Z}(N)\cdot N_v\right\}\mathbf{W}(N)\circ X_v \\ &\quad - 2H_W(N)\left\{\mathbf{W}(N)_u \circ X_u + \mathbf{W}(N)_v \circ X_v\right\} - 2H_W(N)\,\mathbf{W}(N)\circ \Delta X\end{aligned}$$

with the weighted mean curvature

$$H_W(Z) = \frac{\gamma_0}{\sqrt{\det \mathbf{F}_{ZZ}(Z)}}\,.$$

Furthermore there hold

$$\begin{aligned}|\Delta(r^{-1}X)| &\le 2(1+\omega_0)\omega_2|\nabla(r^{-1}X)||\nabla N| + (h_0 r)|\nabla(r^{-1}X)|^2 \\ |\Delta N| &\le 2(1+\omega_0)(h_0 r)^2|\nabla(r^{-1}X)|^2 + (1+\omega_2)|\nabla N|^2 \\ &\quad + 2\left\{(h_0 r)\omega_2 + (1+\omega_0)(h_2 r) + 2(1+\omega_0)^2\omega_2(h_0 r)\right\}|\nabla(r^{-1}X)||\nabla N|.\end{aligned}$$

The second estimate particularly follows from

$$\begin{aligned}|\Delta N| &\le 2|N_u||N_v| + 2\omega_2|N_u||N_v| + 2(1+\omega_0)h_2(|X_u||N_u| + |X_v||N_v|) \\ &\quad + 2h_0\omega_2(|X_u||N_u| + |X_v||N_v|) + 4(1+\omega_0)^2\omega_2 h_0|\nabla X||\nabla N| \\ &\quad + 2(1+\omega_0)h_0^2|\nabla X|^2\,.\end{aligned}$$

11.8 The geometry of immersions of mean curvature type

Recall the elliptic differential operator of second order

$$\Delta_W := \Delta - (\Omega_{11}^1 + \Omega_{22}^1)\frac{\partial}{\partial u} - (\Omega_{11}^2 + \Omega_{22}^2)\frac{\partial}{\partial v}$$

with the classical Laplacian Δ. *A weighted conformally parametrized immersion $X\colon B \to \mathbb{R}^3$ with prescribed weighted mean curvature $H_W(X,Z)$ satisfies*

$$\Delta_W X = 2H_W(X,N)WN \quad \text{in } B.$$

A geometric maximum principle

This system is the basis of the following

Theorem 11.10. *Let the weighted conformally parametrized immersion $X: B \to \mathbb{R}^3$ with prescribed weighted mean curvature $H_W(X,Z)$ be given. Suppose that the smallness condition*

$$h_0 \sup_{(u,v)\in\overline{B}} |X(u,v)| \le 1$$

holds true with the constant

$$h_0 = \sup_{\substack{X\in\mathbb{R}^3 \\ Z\in\mathbb{R}^3\setminus\{0\}}} |H_W(X,Z)|.$$

Then it holds

$$\Delta_W |X(u,v)|^2 \ge 0 \quad \text{in } B.$$

In particular, $|X(u,v)|$ satisfies the geometric maximum principle

$$\sup_{(u,v)\in B} |X(u,v)| = \sup_{(u,v)\in\partial B} |X(u,v)|.$$

Proof. We compute

$$\begin{aligned}\Delta_W |X|^2 &= 2|\nabla X|^2 + 2X\cdot\Delta X - 2(\Omega_{11}^1+\Omega_{11}^2)X\cdot X_u - 2(\Omega_{11}^2+\Omega_{22}^2)X\cdot X_v \\ &= 2|\nabla X|^2 + 4H_W(X,N)WX\cdot N \\ &\ge 2\left\{1-h_0|X|\right\}|\nabla X|^2 .\end{aligned}$$

proving the statement. □

An enclosure theorem for weighted minimal surfaces

As we have seen above, the curvature relation

$$\rho_1(X,N)\kappa_1 + \rho_2(X,N)\kappa_2 = 0$$

with $\rho_1,\rho_2 > 0$ implies $K \le 0$ for the Gauß curvature of a weighted minimal surface. This already implies (see e.g. Sauvigny [141])

Theorem 11.11. *Let the weighted minimal surface $X: B \to \mathbb{R}^3$ be given. Let furthermore $K_r \subset \mathbb{R}^3$ be the smallest closed ball containing the boundary curve $X(\partial B) \subset \mathbb{R}^3$. Then it holds*

$$X(u,v) \subset K_r \quad \text{for all } (u,v) \in B.$$

Proof. Assume that there is an interior point $w_0 \in \mathring{B}$ such that $X(w_0) \notin K_r$. Then due to compactness reasons there is a second point $w_1 \in \mathring{B}$ where the surface touches a sphere $S_R^2 \subset \mathbb{R}^3$ of radius $R > r$. But then it holds $K(w_1) > 0$ contradicting the property $K \leq 0$ in B.

Convex-hull property for weighted minimal surfaces

From $K \leq 0$ we also infer the much stronger

Theorem 11.12. *Let the weighted minimal surface $X\colon B \to \mathbb{R}^3$ be given. Then it holds*

$$X(B) \subset \operatorname{conv} X(\partial B)$$

with $\operatorname{conv} X(\partial B)$ *denoting the convex hull of the boundary curve* $X(\partial B) \subset \mathbb{R}^3$.

Proof. Choose an unit normal vector $Z \in \mathbb{R}^3$ with $|Z| = 1$ and a real number $h \in \mathbb{R}^3$ such that

$$X(\partial B) \subset \mathscr{H}_{Z,h} := \left\{ X \in \mathbb{R}^3 \, : \, Z \cdot X - h \leq 0 \right\}.$$

Now define the function

$$\Phi(u,v) := Z \cdot X(u,v) - h, \quad (u,v) \in B.$$

Then there hold

$$\Delta_W \Phi(u,v) = 0 \quad \text{in } B, \quad \Phi(u,v) \leq 0 \quad \text{on } \partial B.$$

The maximum principle says $\Phi(u,v) \leq 0$ in B, thus $X(B) \subset \mathscr{H}_{Z,h}$. But note

$$\operatorname{conv} X(\partial B) = \bigcap_{Z,h} \mathscr{H}_{Z,h}\,,$$

and it follows $X(B) \subset \mathscr{H}_{Z,h}$ for all half spaces $\mathscr{H}_{Z,h}$. □

These and further maximum principles and enclosure theorems for critical points of anisotropic and inhomogeneous variational problems can be found e.g. in Dierkes [42], [43], or Clarenz [30].

11.9 A curvature estimate

The previous elliptic system together with Heinz' and Sauvigny's methods as briefly described in section 9.4 enable us to prove the following curvature estimate.

Theorem 11.13. *Let the weighted conformally parametrized immersion $X\colon B \to \mathbb{R}^3$ with prescribed weighted mean curvature $H_W(X,Z)$ be given.*

We make the following assumptions:

(A1) Assume that $X = X(u,v)$ *represents a geodesic disc* $\mathfrak{B}_r(X_0)$ *of geodesic radius* $r > 0$ *with center* $X_0 = X(0,0)$.

(A2) Assume that the Dirichlet energy growths quadratically as follows

$$\mathscr{D}[X] \leq d_0 r^2 \tag{11.34}$$

with a real constant $d_0 \in (0,+\infty)$ *independent of r.*

(A3) For real $\lambda = \lambda(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r) > 0$ *there exists* $\delta = \delta(\lambda)$ *such that*

$$|X(w) - X(w_1)| \leq \lambda \quad \textit{and} \quad |N(w) - N(w_1)| \leq \lambda \quad \textit{for all } w \in \overline{B}_\delta(w_0)$$

for arbitrary $w_1 \in B_\delta(w_0)$ *and* $w_0 \in \mathring{B}$ *with* $B_\delta(w_0) \subset\subset \mathring{B}$.

Then it holds the curvature estimate

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \leq \frac{1}{r^2}\,\Theta(d_0, \omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r) + 4(1+\omega_0)^2(2\omega_0 + \omega_0^2)h_0$$

for the principal curvatures κ_1 *and* κ_2 *of the surface X at the origin* $(0,0) \in B$ *with a real constant* $\Theta \in (0,+\infty)$.

Remarks

We have already discussed various situations where some of the assumption from our theorem were already realized. Others are discussed in the following chapter on the second variation. Let us make now some additional remarks.

1. The structure of the right hand side in (11.34) becomes important for our following Bernstein type theorems. In fact, our curvature estimate requires only some positive constant $d_0 > 0$ bounding the Dirichlet energy $\mathscr{D}[X]$.
2. Moduli of continuity for the surface vector can be deduced from geometric maximum principles as we discussed in this and previous chapters. This especially includes

   (i) the convex-hull property for immersions with non-positive Gauss curvature;
   (ii) the enclosure theorems for immersions with nonvanishing weighted or non-weighted mean curvature fields which are small in the sense of

   $$h_0 \sup_{(u,v)\in B} |X(u,v)| < 1.$$

   For these results we refer to the sections 9.3, 10.2 and 11.8.
3. Moduli of continuity for the spherical mapping are discussed in the following chapter on the second variation by means of special classes of surfaces.

An auxiliary result

For proving our curvature estimate we need the following lemma from Sauvigny [141] which characterizes the so-called *plane mapping* of a weighted minimal surface.

Lemma 11.8. *Let the weighted conformally parametrized immersion $X : B \to \mathbb{R}^3$ be given. Then there exists a constant $c_* = c_*(\omega_0) \in (0,1)$ such that the plane mapping*

$$F(u,v) := \big(x^1(u,v), x^2(u,v)\big), \quad (u,v) \in B,$$

satisfies

$$c_*(\omega_0)|\nabla X(u,v)|^2 \le |\nabla F(u,v)|^2 \le |\nabla X(u,v)|^2 \quad \textit{in } B.$$

We would like to remark that it holds

$$\frac{1}{2}|\nabla X|^2 \le |\nabla F|^2 \le |\nabla X|^2$$

using conformal parameters.

Proof. Let as usual $\mathbf{W}(X,Z) = (w_{ij}(X,Z))_{i,j=1,2,3}$. We rewrite the weighted conformality relations

$$\sum_{k,\ell=1}^{3} w_{k\ell}\, x_u^k x_u^\ell = \sum_{k,\ell=1}^{3} w_{k\ell}\, x_v^k x_v^\ell, \quad \sum_{k,\ell=1}^{3} w_{k\ell}\, x_u^k x_v^\ell = 0 \quad \text{in } B$$

into the complex form

$$\sum_{k,\ell=1}^{3} w_{k\ell}\, x_w^k x_w^\ell = 0, \quad w = u + iv, \quad |w| < 1$$

with $x_w = \frac{1}{2}(x_u - ix_v)$. Namely, we note that

$$\begin{aligned}\sum_{k,\ell=1}^{3} w_{k\ell}\, x_w^k x_w^\ell &= \frac{1}{4}\sum_{k,\ell=1}^{3} w_{k\ell}\,(x_u^k - ix_v^k)(x_u^\ell - ix_v^\ell)\\ &= \frac{1}{4}\sum_{k,\ell=1}^{3} (w_{k\ell}\, x_u^k x_u^\ell - w_{k\ell}\, x_v^k x_v^\ell) - \frac{i}{2}\sum_{k,\ell=1}^{3} w_{k\ell}\, x_u^k x_v^\ell,\end{aligned}$$

and the right hand side vanishes identically. From the condition

$$(1+\omega_0)^{-1}|\xi|^2 \le w_{k\ell}\xi^k\xi^\ell \le (1+\omega_0)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3$$

on the weight matrix we infer

$$w_{kk} \ge (1+\omega_0)^{-1} \quad \text{and} \quad |w_{k\ell}| \le c_1(\omega_0), \quad k,\ell = 1,2,3,$$

with a real constant $c_1 \in [0, +\infty)$.

Now let $\varepsilon > 0$ be given. We compute

$$\begin{aligned}(1+\omega_0)^{-1}|x_w^3|^2 \le |w_{33}\,x_w^3 x_w^3| &= \left|\sum_{k,\ell=1}^{3} w_{k\ell}\,x_w^k x_w^\ell - w_{33}\,x_w^3 x_w^3\right| \\ &\le c_1(\omega_0)\Big(|x_w^1|^2+|x_w^2|^2+2|x_w^1||x_w^2|+2|x_w^1||x_w^3|+2|x_w^2||x_w^3|\Big) \\ &\le \left(2+\frac{1}{\varepsilon}\right)c_1(\omega_0)\Big(|x_w^1|^2+|x_w^2|^2\Big)+2\varepsilon c_1(\omega_0)|x_w^3|^2\,.\end{aligned}$$

Thus for arbitrary small $\varepsilon > 0$ we find $c_2 = c_2(\omega_0)$ with the property

$$|x_w^3|^2 \le c_2(\omega_0)(|x_w^1|^2+|x_w^2|^2)$$

from where we conclude

$$|\nabla X|^2 = |\nabla F|^2 + 4|x_w^3|^2 \le \Big\{1+c_2(\omega_0)\Big\}|\nabla F|^2 \quad \text{in } B.$$

The statement follows with $c_* := (1+c_2)^{-1}$. $\square$

Proof of the curvature estimate

Now we come to the proof of the theorem.

1. From the linear dependence of the three weighted fundamental forms

$$N_{u^i}\circ \mathbf{W}(X,N)^{-1}\circ N_{u^j} - 2H_W\,X_{u^i}\cdot N_{u^j} + K\,X_{u^i}\circ \mathbf{W}(X,N)\circ X_{u^j} = 0$$

for $i,j = 1,2$ from section 5.1 we infer

$$N_u\circ \mathbf{W}(X,N)^{-1}\circ N_u + N_v\circ \mathbf{W}(X,N)^{-1}\circ N_v = 2\Big\{2H_W(X,N)^2 - K\Big\}W$$

using weighted conformal parameters and making use of

$$-X_u\cdot N_u - X_v\cdot N_v = L_{11}+L_{22} = 2H_W W.$$

With $\rho_1\kappa_1+\rho_2\kappa_2 = 2H_W$ we compute

$$\rho_1^2\kappa_1^2 + 2\rho_1\rho_2\kappa_1\kappa_2 + \rho_2^2\kappa_2^2 = 4H_W^2$$

or after rearranging

$$\frac{\rho_1}{\rho_2}\,\kappa_1^2 + \frac{\rho_2}{\rho_1}\,\kappa_2^2 = \frac{4}{\rho_1\rho_2}\,H_W^2 - 2\kappa_1\kappa_2 = \frac{4}{\rho_1\rho_2}\,H_W^2 - 2K.$$

Now we can estimate as follows

$$\begin{aligned}(1+\omega_0)|\nabla N|^2 &\geq 4H_W^2 W - 2KW \\ &= 4\left(1-\frac{1}{\rho_1\rho_2}\right)H_W^2 W + \left(\frac{\rho_1}{\rho_2}\,\kappa_1^2 + \frac{\rho_2}{\rho_1}\,\kappa_2^2\right)W \\ &\geq -4(2\omega_0+\omega_0^2)h_0^2 W + \frac{1}{(1+\omega_0)^2}\,(\kappa_1^2+\kappa_2^2)W\end{aligned}$$

where we make us of

$$\frac{1}{1+\omega_0} \leq \rho_1(X,N), \rho_2(X,N) \leq 1+\omega_0\,.$$

Rearranging for the principal curvatures at $(0,0)$ gives

$$\begin{aligned}&\kappa_1(0,0)^2+\kappa_2(0,0)^2 \\ &\quad\leq (1+\omega_0)^3\,\frac{|\nabla N(0,0)|^2}{W(0,0)} + 4(1+\omega_0)^2(2\omega_0+\omega_0^2)h_0^2\,.\end{aligned} \tag{11.35}$$

Thus we must establish an upper bound for the gradient $|\nabla N(0,0)|$ and a lower bound for the area element $W(0,0)$.

For this we essentially use the assumed moduli of continuity: First for an upper gradient bound of the coupled mapping $\mathfrak{X} = (X,N)$, then for a local modulus of projection of the spherical mapping to obtain a lower bound for the area element.

2. The assumed moduli of continuity immediately yield gradient bounds for X and N as follows: Let as usual

$$\mathfrak{X}(w) := \big(r^{-1}X(w), N(w)\big), \quad w \in B,$$

then for given $\lambda > 0$ we find $\delta = \delta(\lambda)$ such that

$$|\mathfrak{X}(w) - \mathfrak{X}(w_1)| \leq 2\lambda \quad \text{for all } w \in B_\delta(w_0)$$

where $w_1 \in B_\delta(w_0)$ chosen arbitrarily within the disc $B_\delta(w_0) \subset\subset \mathring{B}$ with center $w_0 \in B_{1-\nu_0}(0,0)$. The parameter $\nu_0 > 0$ will be fixed in point 6 of the proof. The new mapping

$$\widehat{\mathfrak{X}}(w) := \frac{1}{2\lambda}\Big\{\mathfrak{X}(w) - \mathfrak{X}(w_1)\Big\} : B_\delta(w_0) \longrightarrow \mathbb{R}^6$$

thus satisfies

$$|\widehat{\mathfrak{X}}(w)| \leq 1 \quad \text{and} \quad |\triangle\widehat{\mathfrak{X}}(w)| \leq 2\lambda\Lambda_{\mathfrak{X}}|\nabla\widehat{\mathfrak{X}}|^2 \quad \text{in } B_\delta(w_0)$$

with $\Lambda_{\mathfrak{X}} = \Lambda_{\mathfrak{X}}(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r)$ from (11.31).

Now choose $\lambda = \lambda(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r)$ such that $2\lambda\Lambda_{\mathfrak{X}} \le \frac{1}{2}$. It follows

$$|\Delta\widehat{\mathfrak{X}}(w)| \le \frac{1}{2}|\nabla\mathfrak{X}|^2 \quad \text{in } B_\delta(w_0)$$

with suitable $\delta = \delta(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r)$. Heinz' gradient estimates from section 9.4 ensure the existence of a constant $c_1 \in (0,+\infty)$ with the property

$$\begin{aligned} &|\nabla\mathfrak{X}(w_0)| \le c_1(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r) \\ &\text{for all } w_0 \in \mathring{B}_{1-\nu_0}(0,0). \end{aligned} \tag{11.36}$$

3. This gradient estimate allows us to linearize the system for $r^{-1}X$ as follows

$$\begin{aligned} &|\Delta(r^{-1}X)(w_0)| \le c_2(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r)|\nabla(r^{-1}X)(w_0)| \\ &\text{for all } w_0 \in \mathring{B}_{1-\nu_0}(0,0) \end{aligned}$$

with the constant

$$c_2 := 2\sqrt{2}c_1(1+\omega_0)\sqrt{2(r\omega_1)^2 + 2\omega_2^2} + c_1(h_0 r) \in (0,+\infty).$$

4. Secondly we arrive at the following modulus of projection for the surface: Choose $\nu \in (0, \nu_0/4)$ such that $2\nu c_1 \le 1$ is true. Then it holds

$$|N(w) - N(w_0)| \le 2\nu c_1 \le 1 \quad \text{for all } \mathring{B}_{2\nu}(w_0) \text{ and all } w_0 \in \mathring{B}_{1-2\nu_0}(0,0).$$

5. Now we establish an inequality of Harnack-type for the area element W : For this we choose a fix point $w_0 \in B_{1-2\nu_0}(0,0)$. Assume furthermore $X(w_0) = (0,0,0)$ and $N(w_0) = (0,0,0)$ which can be ensured by rotation and translation. Consider now the plane mapping

$$F(w) := \frac{1}{r}\left(x^1(w_0 + 2\nu w), x^2(w_0 + 2\nu w)\right) : B \longrightarrow \mathbb{R}^2$$

with the number $\nu > 0$ from the previous point of the proof. Due to the foregoing lemma we have

$$\begin{aligned} |\triangle F(w)| &\le 4\nu^2|\triangle(r^{-1}X)(w_0 + 2\nu w)| \\ &\le 4\nu^2 c_2|\nabla(r^{-1}X)(w_0 + 2\nu w)| \\ &\le \frac{2\nu c_2}{\sqrt{c_*}}|\nabla F(w)| \\ &=: c_3(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, \nu_0)|\nabla F(w)| \end{aligned} \tag{11.37}$$

for $w \in \mathring{B}$ with the constant $c_3 := \frac{\nu c_2}{\sqrt{c_*}}$.

From the fourth point of our proof we know

$$J_F(w) > 0 \quad \text{in } \mathring{B}$$

for the Jacobian of the plane mapping $F = F(w)$. As in the proof of our curvature estimate for immersions with prescribed mean curvature fields from section 10.3 we find a constant $c_4 = c_4(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, d_0)$ such that the following Harnack inequality is fulfilled

$$\begin{aligned} |\nabla F(w)|^2 &\le \big(|F_u(w)| + |F_v(w)|\big)^2 \\ &\le c_4^2\big(|F_u(0,0)| + |F_v(0,0)|\big)^{\frac{2}{5}} \\ &\le \sqrt[5]{2}\, c_4^2 |\nabla F(0,0)|^{\frac{2}{5}} \end{aligned} \tag{11.38}$$

for all $w \in \mathring{B}_{\frac{1}{2}}(0,0)$. Furthermore we obtain

$$\begin{aligned} |\nabla F(w)|^2 &\ge \frac{16\nu^4 c_2^2 c_*}{4\nu^2 c_2^2} |\nabla(r^{-1}X)(w_0 + 2\nu w)|^2 \\ &= \frac{4\nu^2 c_*}{r^2} |\nabla X(w_0 + 2\nu w)|^2 \\ &\ge \frac{8\nu^2 c_*}{1+\omega_0} \frac{W(w_0 + 2\nu w)}{r^2} \end{aligned}$$

and analogously

$$|\nabla F(w)|^2 \le \frac{4\nu^2}{r^2} |\nabla X(w_0 + 2\nu w)|^2 \le 8\nu^2(1+\omega_0) \frac{W(w_0 + 2\nu w)}{r^2}. \tag{11.39}$$

Summarizing it follows

$$c_5 \frac{W(w_0 + 2\nu w)}{r^2} \le |\nabla F(w)|^2 \le c_6 \frac{W(w_0 + 2\nu w)}{r^2}$$

with the constants

$$c_5 := \frac{8\nu^2 c_*}{1+\omega_0}, \quad c_6 := 8\nu^2(1+\omega_0).$$

Thus together with (11.38) and (11.39) we conclude

$$c_5 \frac{W(w_0 + 2\nu w)}{r^2} \le |\nabla F(w)|^2 \le \sqrt[5]{2c_6}\, c_4^2 \left(\frac{W(w_0)}{r^2}\right)^{\frac{1}{5}}$$

for all $w \in \mathring{B}_{\frac{1}{2}}(0,0)$.

Rearranging yields the desired Harnack-type inequality for the area element

$$c_7 \left(\frac{W(w)}{r^2} \right)^5 \le \frac{W(w_0)}{r^2} \quad \text{for all } w \in B_\nu(w) \text{ and all } w_0 \in \mathring{B}_{1-2\nu_0}(0,0) \quad (11.40)$$

with the constant

$$c_7(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, d_0) := \frac{1}{2} c_6^{-1} c_5^5 c_4^{-10}.$$

6. Let $\Gamma(B)$ denote the set of all continuous and piecewise differentiable curves $\gamma\colon [0,1] \to B$ with the properties

$$\gamma(0) = (0,0) \quad \text{and} \quad \gamma(1) \in \partial B.$$

Then it holds

$$\inf_{\gamma \in \Gamma(\overline{B})} \int_0^1 \left| \frac{d}{dt} X(\gamma(t)) \right| dt \ge r.$$

As in the proof of the curvature estimate from section 10.3 we can show that there is a point $w^* \in \mathring{B}_{1-\nu_0}$ setting $\nu_0 \ge e^{-4\pi d_0} \in (0,1)$ such that

$$\frac{W(w^*)}{r^2} \ge \frac{1}{16(1+\omega_0)(1-\nu_0)^2} =: c_8(\omega_0, d_0) \qquad (11.41)$$

with the constant $c_8 = c_8(\omega_0, d_0) \in (0, +\infty)$.

7. Now we want to establish a lower bound for the area element. For this purpose we choose a natural number $n \in \mathbb{N}$ sufficiently large such that

$$1 - 2\nu \le n\nu \le 1 - \nu.$$

We define points

$$w_j := \frac{j}{n} w_*, \quad j = 0,1,2,\ldots,n, \; w_* \in \mathring{B}_{1-\nu_0}(0).$$

Then we arrive at

$$|w_j| = \frac{j}{n} |w_*| \le |w_*| \le 1 - \nu_0 \qquad \text{for} \quad j = 0,1,2,\ldots,n$$

which implies

$$|w_{j+1} - w_j| = \tfrac{1}{n} |w_*| \le \tfrac{1-\nu_0}{n} \le \tfrac{1-2\nu}{n} \le \nu \quad \text{for } j = 0,1,2,\ldots,n-1. \qquad (11.42)$$

On account of (11.40) and (11.41) it follows that

$$\frac{W(w_0)}{r^2} \geq c_7 \left(\frac{W(w_1)}{r^2}\right)^5 \geq c_7^{1+5} \left(\frac{W(w_2)}{r^2}\right)^{5^2} \geq c_7^{1+5+5^2+\ldots+5^{n-1}} \left(\frac{W(w_n)}{r^2}\right)^{5^n}$$
$$= c_7^{1+5+5^2+\ldots+5^{n-1}} \left(\frac{W(w_*)}{r^2}\right)^{5^n} \geq c_7^{1+5+5^2+\ldots+5^{n-1}} c_8^{5^n} . \tag{11.43}$$

Together with

$$c_9(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, d_0) := c_7^{1+5+5^2+\ldots+5^{n-1}} c_8^{5^n}$$

we also infer

$$\frac{W(w_0)}{r^2} \geq c_9(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, d_0).$$

8. We collect (11.35), (11.36) and (11.43) to arrive at the estimate

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \leq \frac{1}{r^2}(1+\omega_0)^3 \frac{c_1}{c_9} + 4(1+\omega_0)^2(2\omega_0 + \omega_0^2)h_0^2$$

where $w_0 = (0,0)$. If we finally set

$$\Theta(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, d_0)$$
$$:= (1+\omega_0)^3 \frac{c_1(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r)}{c_9(\omega_0, r\omega_1, \omega_2, h_0 r, h_1 r^2, h_2 r, d_0)}$$

the theorem follows. $\square$

Chapter 12

Crystalline functionals in $\mathbb{R}^{n+2}$

This chapter is devoted to extend our considerations to the analysis of criticical points of crystalline functionals in Euclidean spaces of higher dimensions. We put some important results on Plateau's problem for such parametric functionals together, elaborate on their first variations and discuss their relationships to our variational problems so far.

12.1 Crystalline functionals. Remarks on Plateau's problem

In generalization of our earlier discussions from the foregoing chapters we now consider *inhomogeneous and anisotropic Lagrangian densities* of the form

$$F \in C^{5+\alpha}(\mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}) \in C^0(\mathbb{R}^n \times \mathbb{R}^N, \mathbb{R})$$

and variational problems for the *functional of crystalline type*

$$\mathscr{B}[X] := \iint\limits_B F(X, X_u \wedge X_v)\, dudv \longrightarrow \text{extr!} \tag{12.1}$$

As discussed in the foregoing chapter, such Lagrangians $F(X, X_u \wedge X_v)$ arise from the theory of crystall growth; see the original work of G. Wulff in [174], or the later contributions of J.E. Taylor, e.g. [153], [154] in the context of geometric measure theory.

Before we go into calculations we want to recall some history of the so-called *Plateau problem.* In its original form it aks for finding *a minimal surface spanning a given curve in space which is critical for the area functional*

$$\mathscr{A}[X] = \iint\limits_B |X_u \times X_v|\, dudv.$$

Garnier [69], Douglas [47] and Radó [133] independently succeeded in the construction of such surfaces *minimizing the area among all admissable immersions spanning the same boundary curve.* Owing to his work, Jesse Douglas was granted with the first Fields Medal in 1936.

It was also Douglas in [48] and [49] who first succeeded on Plateau's problem on multiple connected domains which particularly involves minimal surfaces spanning more than one boundary curve.

For Plateau's problem in Riemannian or hyperbolic spaces we want to refer to classical works of Lonseth [117] from 1942 and Morrey [121] from 1948.

The mathematician's attention to anisotropic and inhomogenous variational problems was already drawn in the early fifties, see e.g. Cesari [24], Danskin [41], Jenkins [99], or Morrey [122], [123]. Using methods from geometric measure theory, B. White in [170], [171] proved existence of smooth embeddings solving this "generalized" Plateau problem assuming the boundary curve is contained in the boundary of a convex body. For a good introduction into the powerful methods of geometric measure theory we want to refer the reader to Almgren [2].

In a series of papers, this general Plateau problems was revived by S. Hildebrandt and H. von der Mosel; see [86], [87], [88], [89], [90], [91]. The Douglas problem for inhomogenous and anisotropic parametric functionals, asking for the construction of minimizers of higher topological type, was considered by Kurzke and von der Mosel [110].

Existence and regularity of minimizers

The following existence and regularity results for minimizers of the general problem 12.1 are taken from Hildebrandt and von der Mosel [87], Theorem 1.4 and 1.5.

Theorem 12.1. *If the Lagrangian density $F\in C^0(\mathbb{R}^n\times\mathbb{R}^N,\mathbb{R})$ is*

- *positive homogeneous* $F(X,\lambda Z)=\lambda F(X,Z)$ *for all* $\lambda>0$;
- *positive definite* $m_1|Z|\le F(X,Z)\le m_2|Z|$ *with* $0<m_1\le m_2<\infty$;
- *convex* $F(X,\lambda Z_1+\mu Z_2)\le\lambda F(X,Z_1)+\mu F(X,Z_2)$ *for all* $\lambda,\mu>0,\ \lambda+\mu=1$

for all $X\in\mathbb{R}^{n+2}$ and all $Z,Z_1,Z_2\in\mathbb{R}^N$, then there exits a minimizer $X\colon B\to\mathbb{R}^{n+2}$ of $\mathscr{B}[X]$ in the following class of immersions

$$\mathscr{C}(\Gamma):=\Big\{X\in H^{1,2}(\mathring{B},\mathbb{R}^{n+2}):\ X|_{\partial B}\ \textit{is a continuous, weakly monotonic mapping from}\ \partial B\ \textit{to}\ \Gamma\Big\}$$

which is conformally parametrized, i.e. $X\in\mathscr{C}(\Gamma)$ satisfies

$$\mathscr{B}[X]=\inf_{\widetilde{X}\in\mathscr{C}(\Gamma)}\mathscr{B}[\widetilde{X}]$$

as well as the conformality relations

$$X_u\cdot X_u=X_v\cdot X_v\,,\quad X_u\cdot X_v=0\quad\textit{almost everywhere on } B.$$

Furthermore, every conformally parametrized minimizer X of $\mathscr{B}[X]$ in the class $\mathscr{C}(\Gamma)$ satisfies

$$X \in C^0(B,\mathbb{R}^n) \cap C^{0,\gamma}(\mathring{B},\mathbb{R}^n) \quad \textit{with } \gamma = \frac{m_1}{m_2}$$

as well as the Morrey growth condition

$$\iint_{B_r(w_0)} |\nabla X|^2 \, du dv \le \left(\frac{r}{R}\right)^{2\gamma} \iint_{B_R(w_0)} |\nabla X|^2 \, du dv$$

for any $w_0 \in B$ and $0 < r \le R \le R_0 := 1 - |w_0|$.

To establish *boundary regularity* we need the concept of a *perfect dominance function.* For this purpose let

$$f(X,P) := F(X, P_1 \wedge P_2)$$

denote the *associated Lagrangian* to the Lagrangian $F(X,P)$.

Definition 12.1. (Hildebrandt, von der Mosel [89], Definition 1)

(i) Let $F(X,Z)$ be given with the associated Lagrangian $f(X,P)$. Then a function

$$G\colon \mathbb{R}^{n+2} \times \mathbb{R}^{2(n+2)} \longrightarrow \mathbb{R}$$

is said to be a *dominance function for F* if it is continous and satisfies the following two conditions

$$f(X,P) \le G(X,Z) \quad \text{for any } (X,Z) \in \mathbb{R}^{n+2} \times \mathbb{R}^{2(n+2)} \qquad (D1)$$

as well as

$$f(X,P) = G(X,P) \quad \text{if and only if } |p_1|^2 = |p_2|^2, \; p_1 \cdot p_2 = 0. \qquad (D2)$$

(ii) A dominance function G of F is called *quadratic* if

$$G(X,\lambda P) = \lambda^2 G(X,P) \quad \text{for all } \lambda > 0, \; (X,P) \in \mathbb{R}^{n+2} \times \mathbb{R}^{2(n+2)}, \qquad (D3)$$

and it is said to be *positive definite* if there are two numbers $0 < \mu_1 \le \mu < \infty$ with

$$\mu_1 |P|^2 \le G(X,P) \le \mu_2 |P|^2 \quad \text{for any } (X,P) \in \mathbb{R}^{n+2} \times \mathbb{R}^{2(n+2)}. \qquad (D4)$$

(iii) A function

$$G \in C^0(\mathbb{R}^{n+2} \times \mathbb{R}^{2(n+2)}) \cap C^2(\mathbb{R}^{n+2} \times \mathbb{R}^{2(n+2)} \setminus \{0\})$$

is called a *perfect dominance function for F* if it satisfies the foregoing conditions (D1)–(D4), and if for any $R_0 > 0$ there is a constant $\lambda_G(R_0) > 0$ such that

$$\xi \circ \mathbf{G}_{PP}(X,P) \circ \xi \le \lambda_G(R_0) |\xi|^2 \quad \text{for } |X| \le R_0 \text{ and } P, \xi \in \mathbb{R}^{2(n+2)}, \; P \ne 0.$$

Before we state the boundary regularity theorem we give two examples (see Hildebrandt and von der Mosel [89]).

1. The integrand
$$D(P) := \frac{1}{2}|P|^2 = \frac{1}{2}|P_1|^2 + \frac{1}{2}|P_2|^2$$
of the Dirichlet energy represents a perfect dominance function for the area integrand $|P_1 \wedge P_2|$.
2. The integrand
$$G(X,P) := \frac{1}{2}|P|^2 + Q(X)\cdot(P_1 \wedge P_2)$$
is a perfect dominance function for $E(X,Z) := |Z| + Q(X)\cdot Z$.

The existence of perfect dominance functions for a large class of Lagrangians is the contents of the next result from Hildebrandt and von der Mosel [89].

Theorem 12.2. *Let $F^* \in C^0(\mathbb{R}^{n+2}\times\mathbb{R}^N)\cap C^2(\mathbb{R}^{n+2}\times\mathbb{R}^N\setminus\{0\})$ be positive homogeneous, positive definite with constants m_1^* and m_2^*, and convex, and assume that*

$$|Z|\,\xi\circ\mathbf{F}^*_{ZZ}(X,Z)\circ\xi \geq \lambda_{F^*}(R_0)\big|\xi - |Z|^{-2}(Z\cdot\xi)Z\big|^2$$

with the ellipticity constant

$$\lambda^* := \inf_{R_0\in(0,\infty]} \lambda_{F^*}(R_0) > 0.$$

Then for
$$k > k_0 := 2\big[m_2^* - \min\{\lambda^*, m_1^*/2\}\big]$$
the parametric Lagrangian

$$F(X,Z) := kA(Z) + F^*(X,Z)$$

possesses a perfect dominance function.

Now comes Hildebrandt's and von der Mosel's boundary regularity result from [88], Theorem 1.1.

Theorem 12.3. *Suppose that F is of class $C^2(\mathbb{R}^{n+2}\times\mathbb{R}^N\setminus\{0\},\mathbb{R})$ and that it is positive homogeneous, positive definite and convex. Suppose also that F possesses a perfect dominance function G, and that the boundary curve Γ is of class C^4. Then there is some $\alpha\in(0,1)$ such that any conformally parametrized minimizer X of $\mathscr{B}[X]$ is of class $H^{2,2}(\mathring{B},\mathbb{R}^{n+2})\cap C^{1,\alpha}(B,\mathbb{R}^{n+2})$ and satisfies*

$$\|X\|_{H^{2,2}(\mathring{B},\mathbb{R}^{n+2})} + \|X\|_{C^{1,\alpha}(B,\mathbb{R}^{n+2})} \leq c(\Gamma,F)$$

where the number $c(\Gamma,F)$ depends only on Γ and F.

Further remarks on the literature

Sauvigny in [141] established curvature estimates for *F-minimal surfaces* which may arise as critical points for the functional

$$\mathscr{F}[X] = \iint_B F(X_u \times X_v)\,dudv$$

in case of one codimension; see also Fröhlich [62] for further developments.

Various geometric and analytical results for critical points of $\mathscr{F}[X]$ in one codimension were particularly established in Lin [115], Räwer [134], Clarenz [31], Clarenz and von der Mosel [32]. Anisotropic variational problems with applications to rotationally symmetric Delaunay-type surfaces in case $n = 1$ of one codimension can be found in Koiso and Palmer [107], [108].

Moreover, Winklmann in [172], [173] established integral and pointwise curvature estimates for m-dimensional manifolds in $\mathbb{R}^{m+1}$ which are critical for functionals similar to (12.1). The methods he used there go essentially back to Ecker [50] and Schoen, Simon, Yau [145].

For considerations on anisotropic mean curvature flows we want to refer to Clutterbuck [33], or Pozzi [132] for numerical investigations.

12.2 Parameter invariance

Before we are going to compute the first variation of $\mathscr{B}[X]$ we will establish a necessary and a sufficient condition for the functional to be independent of the choice of the parameter system.

Proposition 12.1. *$\mathscr{B}[X]$ is invariant w.r.t. a parameter transformations from class $\mathfrak{P}$ and of the chosen parameter domain if and only it holds*

$$F(X,\lambda Z) = \lambda F(X,Z) \quad \textit{for all real } \lambda > 0. \tag{12.2}$$

We omit the proof for it works exactly as our calcululations from section 11.2.

12.3 Euler's homogeneity relations

As in the previous chapter we introduce the following notations.

Definition 12.2. We set

$$\begin{aligned} F_X(X,Z) &:= (F_{x^1}(X,Z),\ldots,F_{x^n}(X,Z)) \in \mathbb{R}^n\,, \\ F_Z(X,Z) &:= (F_{z^1}(X,Z),\ldots,F_{z^N}(X,Z)) \in \mathbb{R}^N\,, \\ \mathbf{F}_{XX}(X,Z) &:= (F_{x^ix^j}(X,Z))_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}\,, \\ \mathbf{F}_{XZ}(X,Z) &:= (F_{x^iz^j}(X,Z))_{i=1,\ldots,n,\,j=1,\ldots,N} \in \mathbb{R}^{n\times N}\,, \\ \mathbf{F}_{ZZ}(X,Z) &:= (F_{z^iz^j}(X,Z))_{i,j=1,\ldots,N} \in \mathbb{R}^{N\times N} \end{aligned}$$

with the abbreviation $N := \binom{n}{2}$.

Now the homogeneity condition (12.2) allows us to infer the following properties in the same way we proceeded in the foregoing chapter.

Proposition 12.2. *The vector $F_Z(X,Z)$ is positive-homogeneous of degree 0 w.r.t. Z, the matrix $\mathbf{F}_{ZZ}(X,Z)$ is positive-homogeneous of degree -1 w.r.t. Z. Furthermore their hold the homogeneity relations*

$$F_Z(X,Z)\cdot Z = F(X,Z), \quad \mathbf{F}_{ZZ}(X,Z)\circ Z = 0, \quad \mathbf{F}_{XZ}(X,Z)\circ Z = F_X(X,Z)$$

for all $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^n \setminus \{0\}$.

12.4 The first variation

We compute the Euler-Lagrange equations of the general parametric functional $\mathscr{B}[X]$. For this purpose it is necessary to recall the Grassmann-type formalism

$$X \wedge Y = \sum_{1\le i<j\le n} (x^iy^j - x^jy^i)e_i \wedge e_j$$

introduced in chapter 6.

Theorem 12.4. *The Euler-Lagrange equations of $\mathscr{B}[X]$ are*

$$\begin{aligned} &F_X(X,\mathscr{N})\cdot N_\sigma W + (N_\sigma \wedge X_v)\circ \mathbf{F}_{ZX}(X,\mathscr{N})\circ X_u + (X_u\wedge N_\sigma)\circ \mathbf{F}_{ZX}(X,\mathscr{N})\circ X_v \\ &\quad = -(N_\sigma\wedge X_v)\circ \mathbf{F}_{ZZ}(X,\mathscr{N})\circ \mathscr{N}_u - (X_u\wedge N_\sigma)\circ \mathbf{F}_{ZZ}(X,\mathscr{N})\circ \mathscr{N}_v \end{aligned} \tag{12.3}$$

in B for all $\sigma = 1,\ldots,n$ with the Grassmann-type vector

$$\mathscr{N} := \frac{X_u \wedge X_v}{|X_u\wedge X_v|}$$

from (6.3).

Proof. Consider the variation

$$\widetilde{X} = X + \varepsilon \sum_{\sigma=1}^{n} \varphi_\sigma N_\sigma$$

with test functions $\varphi_i \in C_0^\infty(B,\mathbb{R})$ and a real $\varepsilon \in (-\varepsilon_0, +\varepsilon_0)$. First we calculate

$$\begin{aligned}
\widetilde{X}_u \wedge \widetilde{X}_v &= \left(X_u + \varepsilon \sum_{\sigma=1}^{n} \left\{ \varphi_{\sigma,u} N_\sigma + \varphi N_{\sigma,u} \right\} \right) \wedge \left(X_v + \varepsilon \sum_{\sigma=1}^{n} \left\{ \varphi_{\sigma,v} N_\sigma + \varphi_\sigma N_{\sigma,v} \right\} \right) \\
&= X_u \wedge X_v + \varepsilon \sum_{\sigma=1}^{n} \left\{ \varphi_{\sigma,u} X_u \wedge N_\sigma + \varphi_{\sigma,v} N_\sigma \wedge X_v \right\} \\
&\quad + \varepsilon \sum_{\sigma=1}^{n} \left\{ X_u \wedge N_{\sigma,v} + N_{\sigma,u} \wedge X_v \right\} \varphi_\sigma + o(\varepsilon).
\end{aligned}$$

We compute the derivative

$$\frac{\partial}{\partial \varepsilon} \mathscr{B}[\widetilde{X}] = \iint\limits_B \left\{ F_X(\widetilde{X}, \widetilde{X}_u \wedge \widetilde{X}_v) \cdot \frac{\partial}{\partial \varepsilon} \widetilde{X} + F_X(\widetilde{X}, \widetilde{X}_u \wedge \widetilde{X}_v) \cdot \frac{\partial}{\partial \varepsilon} (\widetilde{X}_u \wedge \widetilde{X}_v) \right\} dudv$$

using

$$\frac{\partial}{\partial \varepsilon} \widetilde{X} \big|_{\varepsilon=0} = \sum_{\sigma=1}^{n} \varphi_\sigma N_\sigma$$

as well as

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} (\widetilde{X}_u \wedge \widetilde{X}_v) \big|_{\varepsilon=0} &= \sum_{\sigma=1}^{n} \left\{ \varphi_{\sigma,u} X_u \wedge N_\sigma + \varphi_{\sigma,v} N_\sigma \wedge X_v \right\} \\
&\quad + \sum_{\sigma=1}^{n} \left\{ X_u \wedge N_{\sigma,v} + N_{\sigma,u} \wedge X_v \right\} \varphi_\sigma .
\end{aligned}$$

Taking the Euler homogeneity conditions into account we infer at $\varepsilon = 0$

$$\begin{aligned}
\delta \mathscr{B}[X] &= \sum_{\sigma=1}^{n} \iint\limits_B F_X(X, X_u \wedge X_v) \cdot N_\sigma \, \varphi_\sigma \, dudv \\
&\quad + \sum_{\sigma=1}^{n} \iint\limits_B F_Z(X, X_u \wedge X_v) \cdot (X_u \wedge N_{\sigma,v} + N_{\sigma,u} \wedge X_v) \varphi_\sigma \, dudv \\
&\quad + \sum_{\sigma=1}^{n} \iint\limits_B F_Z(X, X_u \wedge X_v) \cdot (X_u \wedge N_\sigma \varphi_{\sigma,v} + N_\sigma \wedge X_v \varphi_{\sigma,u}) \, dudv \\
&= \ldots
\end{aligned}$$

$$\begin{aligned}
\ldots = &\sum_{\sigma=1}^{n} \iint\limits_{B} F_X(X,\mathscr{N})\cdot N_\sigma W\varphi_\sigma\,dudv \\
&+\sum_{\sigma=1}^{n} \iint\limits_{B} \operatorname{div}\Big(F_Z(X,\mathscr{N})\cdot(N_\sigma\wedge X_v)\varphi_\sigma, F_Z(X,\mathscr{N})\cdot(X_u\wedge N_\sigma)\varphi_\sigma\Big)\,dudv \\
&-\sum_{\sigma=1}^{n} \iint\limits_{B} \big\{\mathbf{F}_{ZX}(X,\mathscr{N})\cdot X_u\big\}\cdot(N_\sigma\wedge X_v)\varphi_\sigma\,dudv \\
&-\sum_{\sigma=1}^{n} \iint\limits_{B} \big\{\mathbf{F}_{ZX}(X,\mathscr{N})\cdot X_v\big\}\cdot(X_u\wedge N_\sigma)\varphi_\sigma\,dudv \\
&-\sum_{\sigma=1}^{n} \iint\limits_{B} \big\{\mathbf{F}_{ZZ}(X,\mathscr{N})\cdot \mathscr{N}_u\big\}\cdot(N_\sigma\wedge X_v)\varphi_\sigma\,dudv \\
&-\sum_{\sigma=1}^{n} \iint\limits_{B} \big\{\mathbf{F}_{ZZ}(X,\mathscr{N})\cdot \mathscr{N}_v\big\}\cdot(X_u\wedge N_\sigma)\varphi_\sigma\,dudv.
\end{aligned}$$

The integral over the divergence vanishes due $\varphi_\sigma = 0$. This proves the theorem. □

12.5 Minimal surfaces

In the special case $F(X,Z) = |Z|$ we have

$$F_X(X,Z) \equiv 0, \quad \mathbf{F}_{XZ}(X,Z) \equiv \mathbf{0}, \quad \mathbf{F}_{ZZ}(X,Z) \equiv \mathbb{E}^N$$

with the N-dimensional unit matrix $\mathbb{E}^N$. Thus we must evaluate the system

$$(N_\sigma\wedge X_v)\cdot\mathscr{N}_u + (X_u\wedge N_\sigma)\cdot\mathscr{N}_v = 0 \quad \text{for all } \sigma = 1,\ldots,n.$$

To this end we introduce conformal parameters $(u,v)\in B$. Then

$$\left\{\frac{X_u}{\sqrt{W}}, \frac{X_v}{\sqrt{W}}, N_1,\ldots,N_n\right\}$$

as well as the set

$$\big\{\mathscr{N}, \mathscr{X}_{11},\ldots,\mathscr{X}_{1n},\mathscr{X}_{21},\ldots,\mathscr{X}_{2n},\mathscr{N}_{11},\ldots,\mathscr{N}_{1n},\mathscr{N}_{23},\ldots,\mathscr{N}_{n-1,n}\big\}$$

consisting of the $N = \binom{n}{2}$ unit normal vectors

$$\mathscr{N} := \frac{X_u\wedge X_v}{W}, \quad \mathscr{X}_{i1} := \frac{X_{u^i}\wedge N_1}{\sqrt{W}}, \ldots, \mathscr{X}_{in} := \frac{X_{u^i}\wedge N_n}{\sqrt{W}}, \quad \mathscr{Z}_{ij} := N_i\wedge N_j$$

form orthornormal frames.

The Euler-Lagrange equations (12.3) take the simpler form

$$\mathscr{X}_{1\sigma} \cdot \mathscr{N}_v - \mathscr{X}_{2\sigma} \cdot \mathscr{N}_u = 0 \quad \text{in } B. \tag{12.4}$$

Next we recall the Grassmann-Weingarten equations (6.4)

$$\mathscr{N}_{u^i} = -\sum_{m=1}^{2} \sum_{\vartheta=1}^{n} \mathscr{L}_i^{m\vartheta} \mathscr{X}_{m\vartheta}$$

with the $\mathscr{G}$-fundamental form $\mathscr{L}_i^{m\vartheta} = -\mathscr{N}_{u^i} \cdot \mathscr{X}_{m\vartheta}$. In terms of these coefficients, equation (12.4) can be written as

$$\mathscr{L}_2^{1\sigma} - \mathscr{L}_1^{2\sigma} = 0 \quad \text{in } B$$

for all $\sigma = 1, \ldots, n$.

We want to prove that these n equations are equivalent to $H_\sigma \equiv 0$ for the mean curvatures H_σ, $\sigma = 1, \ldots, n$, along the unit normal vectors N_σ.

First, from $(N_\sigma \wedge X_v) \cdot \mathscr{N} = 0$ and $(X_u \wedge N_\sigma) \cdot \mathscr{N} = 0$ there follow after differentiation

$$\begin{aligned} (N_\sigma \wedge X_v) \cdot \mathscr{N}_u &= (N_{\sigma,u} \wedge X_v + N_\sigma \wedge X_{uv}) \cdot \mathscr{N}\,, \\ (X_u \wedge N_\sigma) \cdot \mathscr{N}_v &= (X_{uv} \wedge N_\sigma + X_u \wedge N_{\sigma,v}) \cdot \mathscr{N}\,. \end{aligned}$$

The Weingarten equations and the Gauss equations imply

$$\begin{aligned} (N_{\sigma,u} \wedge X_v) \cdot \mathscr{N} &= -L_{\sigma,11} \mathscr{N} \cdot \mathscr{N} - \sum_{\omega=1}^{n} T_{\sigma,1}^{\omega} \mathscr{X}_{2\omega} \cdot \mathscr{N} = -L_{\sigma,11}\,, \\ (X_u \wedge N_{\sigma,v}) \cdot \mathscr{N} &= -L_{\sigma,22} \mathscr{N} \cdot \mathscr{N} + \sum_{\omega=1}^{n} T_{\sigma,2}^{\omega} \mathscr{X}_{1\omega} \cdot \mathscr{N} = -L_{\sigma,22} \end{aligned}$$

as well as

$$\begin{aligned} (N_\sigma \wedge X_{uv}) \cdot \mathscr{N} \; &= \; -\frac{W_v}{2W} \mathscr{X}_{1\sigma} \cdot \mathscr{N} - \frac{W_u}{2W} \mathscr{X}_{2\sigma} \cdot \mathscr{N} + \sum_{\omega=1}^{n} L_{\omega,12} \mathscr{Z}_{\sigma\omega} \cdot \mathscr{N} \; = \; 0, \\ (X_{uv} \wedge N_\sigma) \cdot \mathscr{N} \; &= \; 0. \end{aligned}$$

Thus it follows

$$\begin{aligned} 0 \; &= \; (X_u \wedge N_\sigma) \cdot \mathscr{N}_v + (N_\sigma \wedge X_v) \cdot \mathscr{N}_u \; = \; -(X_u \wedge N_{\sigma,v}) \cdot \mathscr{N} - (N_{\sigma,u} \wedge X_v) \cdot \mathscr{N} \\ &= \; L_{\sigma,22} + L_{\sigma,22} \; = \; 2 H_\sigma W. \end{aligned}$$

Critical points of the area functional have vanishing mean curvature vector.

12.6 The problem for weighted minimal surfaces

Consider finally a critical point $X:\ B \to \mathbb{R}^{n+2}$ of the anisotropic variational problem

$$\iint_B F(\mathscr{N})\,dudv \longrightarrow \text{extr!}$$

Then $X:\ B \to \mathbb{R}^{n+2}$ satisfies the Euler-Lagrange system

$$(N_\sigma \wedge X_v) \circ \mathbf{F}_{ZZ}(\mathscr{N}) \circ \mathscr{N}_u + (X_u \wedge N_\sigma) \circ \mathbf{F}_{ZZ}(\mathscr{N}) \circ \mathscr{N}_v = 0 \quad \text{in } B.$$

It remains open how we can express this system in terms of a weight matrix $\mathbf{W}(X,Z)$ as discussed in chapter 4.

Chapter 13
The second variation

In this chapter we compute the second variation of some selected geometric functionals and define stability and μ-stability. We also discuss various criteria which ensure μ-stability. For this purpose we recall some elementary facts on eigenvalue methods on the sphere. We conclude this chapter with considering the second variation for non-parametric problems.

13.1 Minimal surfaces

The second variation

We start with the following

Theorem 13.1. *Let $N=(N_1,\ldots,N_n)$ be an ONF for the conformally parametrized minimal surface $X\colon B\to\mathbb{R}^{n+2}$. Then the second variation*

$$\delta^2_{N_{\widehat{\gamma}}}\mathscr{A}[X;\varphi]:=\frac{\partial^2}{\partial\varepsilon^2}\,\mathscr{A}[X+\varepsilon\varphi N_{\widehat{\gamma}}]\Big|_{\varepsilon=0}$$

of the area functional $\mathscr{A}[X]$ w.r.t. perturbations

$$\widetilde{X}(u,v)=X(u,v)+\varepsilon\varphi(u,v)N_{\widehat{\gamma}}(u,v),$$

where

$$N_{\widehat{\gamma}}(u,v)=\sum_{\sigma=1}^n\widehat{\gamma}^\sigma(u,v)N_\sigma(u,v),\qquad\sum_{\sigma=1}^n\widehat{\gamma}^\sigma(u,v)^2=1,$$

$\widehat{\gamma}=(\widehat{\gamma}^1,\ldots,\widehat{\gamma}^n)$, $\varphi\in C_0^\infty(B,\mathbb{R})$ and real $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ is given by

$$\begin{aligned}\delta^2_{N_{\widehat{\gamma}}}\mathscr{A}[X;\varphi]\,=&\iint\limits_B(|\nabla\varphi|^2+2K_{\widehat{\gamma}}W\varphi^2)\,dudv\\&+\iint\limits_B\sum_{\sigma=1}^n\left\{\left(\widehat{\gamma}_u^\sigma+\sum_{\omega=1}^n\widehat{\gamma}^\omega T_{\omega,1}^\sigma\right)^2+\left(\widehat{\gamma}_v^\sigma+\sum_{\omega=1}^n\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right)^2\right\}\varphi^2\,dudv\end{aligned}$$

with the Gaussian curvature $K_{\widehat{\gamma}}:=K_{N_{\widehat{\gamma}}}$ along $N_{\widehat{\gamma}}$.

In particular, if the ONF N is free of torsion then

$$\delta^2_{N_{\widehat{\gamma}}}\mathscr{A}[X] = \iint\limits_B (|\nabla\varphi|^2 + |\nabla\widehat{\gamma}|^2\varphi^2 + 2K_{\widehat{\gamma}}W\varphi^2)\,dudv$$

$$\textit{with the setting}\quad |\nabla\widehat{\gamma}|^2 = \sum_{\sigma=1}^{n}\left\{(\widehat{\gamma}_u^\sigma)^2 + (\widehat{\gamma}_v^\sigma)^2\right\}.$$

Proof. From the proof of Theorem 9.1 we know

$$\delta_{N_{\widehat{\gamma}}}g_{11} = -2\varphi L_{\widehat{\gamma},11}\,,\quad \delta_{N_{\widehat{\gamma}}}g_{12} = -2\varphi L_{\widehat{\gamma},12}\,,\quad \delta_{N_{\widehat{\gamma}}}g_{22} = -2\varphi L_{\widehat{\gamma},22}$$

with the coefficients $L_{\widehat{\gamma},ij} = N_{\widehat{\gamma}}\cdot X_{u^iu^j}$. We additionally infer

$$\delta^2_{N_{\widehat{\gamma}}}g_{11} = 2\varphi_u^2 + 2\varphi^2 N^2_{\widehat{\gamma},u}\,,\quad \delta^2_{N_{\widehat{\gamma}}}g_{12} = 2\varphi_u\varphi_v + 2\varphi^2 N_{\widehat{\gamma},u}\cdot N_{\widehat{\gamma},v}\,,$$
$$\delta^2_{N_{\widehat{\gamma}}}g_{22} = 2\varphi_v^2 + 2\varphi N^2_{\widehat{\gamma},v}\,.$$

Using conformal parameters we rewrite these identities as follows

$$\delta^2_{N_{\widehat{\gamma}}}g_{11} = 2\varphi_u^2 + \frac{2}{W}\,\varphi^2\left\{L^2_{\widehat{\gamma},11} + L^2_{\widehat{\gamma},12}\right\} + 2\varphi^2\sum_{\sigma=1}^{n}\left(\widehat{\gamma}_u^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,1}\right)^2,$$

$$\delta^2_{N_{\widehat{\gamma}}}g_{12} = 2\varphi_u\varphi_v + \frac{2}{W}\,\varphi^2(L_{\widehat{\gamma},11} + L_{\widehat{\gamma},22})L_{\widehat{\gamma},12}$$
$$+2\varphi^2\sum_{\sigma=1}^{n}\left(\widehat{\gamma}_u^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,1}\right)\left(\widehat{\gamma}_v^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,2}\right),$$

$$\delta^2_{N_{\widehat{\gamma}}}g_{22} = 2\varphi_v^2 + \frac{2}{W}\,\varphi^2\left\{L^2_{\widehat{\gamma},12} + L^2_{\widehat{\gamma},22}\right\} + 2\varphi^2\sum_{\sigma=1}^{n}\left(\widehat{\gamma}_v^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,2}\right)^2.$$

Namely the Weingarten equations (2.11) imply

$$N_{\widehat{\gamma},u} = -\frac{L_{\widehat{\gamma},11}}{W}X_u - \frac{L_{\widehat{\gamma},12}}{W}X_v + \sum_{\sigma=1}^{n}\left(\widehat{\gamma}_u^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,1}\right)N_\sigma\,,$$

$$N_{\widehat{\gamma},v} = -\frac{L_{\widehat{\gamma},12}}{W}X_u - \frac{L_{\widehat{\gamma},22}}{W}X_v + \sum_{\sigma=1}^{n}\left(\widehat{\gamma}_v^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,2}\right)N_\sigma$$

and therefore

$$N^2_{\widehat{\gamma},u} = \frac{L^2_{\widehat{\gamma},11} + L^2_{\widehat{\gamma},12}}{W} + \sum_{\sigma=1}^{n}\left(\widehat{\gamma}_u^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T^\sigma_{\omega,1}\right)^2,$$

$$N_{\widehat{\gamma},u}\cdot N_{\widehat{\gamma},v} = \frac{L_{\widehat{\gamma},11}+L_{\widehat{\gamma},22}}{W}\,L_{\widehat{\gamma},12}+\sum_{\sigma=1}^{n}\left(\widehat{\gamma}_u^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,1}^\sigma\right)\left(\widehat{\gamma}_v^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right),$$

$$N_{\widehat{\gamma},v}^2 = \frac{L_{\widehat{\gamma},12}^2+L_{\widehat{\gamma},22}^2}{W}+\sum_{\sigma=1}^{n}\left(\widehat{\gamma}_v^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right)^2.$$

This shows the above representations for the $\delta_{N_{\widehat{\gamma}}}^2 g_{ij}$. Next we compute

$$\begin{aligned}&2W\delta_{N_{\widehat{\gamma}}}^2W+2(\delta_{N_{\widehat{\gamma}}}W)^2\\&\quad= g_{11}\delta_{N_{\widehat{\gamma}}}^2g_{22}+2\delta_{N_{\widehat{\gamma}}}g_{11}\delta_{N_{\widehat{\gamma}}}g_{22}+g_{22}\delta_{N_{\widehat{\gamma}}}^2g_{11}-2g_{12}\delta_{N_{\widehat{\gamma}}}^2g_{12}-2(\delta_{N_{\widehat{\gamma}}}g_{12})^2,\end{aligned}$$

i.e. using conformal parameters we arrive at

$$\delta_{N_{\widehat{\gamma}}}^2W = \frac{1}{2}\,\delta_{N_{\widehat{\gamma}}}^2g_{11}+\frac{1}{2}\,\delta^2g_{22}+\frac{1}{W}\,\delta_{N_{\widehat{\gamma}}}g_{11}\delta_{N_{\widehat{\gamma}}}g_{22}-\frac{1}{W}\,(\delta_{N_{\widehat{\gamma}}}g_{12})^2-\frac{1}{W}\,(\delta_{N_{\widehat{\gamma}}}W)^2.$$

Inserting the identities for $\delta_{N_{\widehat{\gamma}}}g_{ij}$ and $\delta_{N_{\widehat{\gamma}}}^2g_{ij}$ previously obtained, together with

$$\delta_{N_{\widehat{\gamma}}}W = -2H_{\widehat{\gamma}}W\varphi$$

from section 9.1, gives us

$$\begin{aligned}\delta_{N_{\widehat{\gamma}}}^2W &= |\nabla\varphi|^2-4H_{\widehat{\gamma}}^2W\varphi^2+\frac{\varphi^2}{W}\,(L_{\widehat{\gamma},11}^2+2L_{\widehat{\gamma},12}^2+L_{\widehat{\gamma},22}^2)+\frac{4\varphi^2}{W}\,(L_{\widehat{\gamma},11}L_{\widehat{\gamma},22}-L_{\widehat{\gamma},12}^2)\\&\quad+\varphi^2\sum_{\sigma=1}^{n}\left\{\left(\widehat{\gamma}_u^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,1}^\sigma\right)^2+\left(\widehat{\gamma}_v^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right)^2\right\}.\end{aligned}$$

But it holds

$$\begin{aligned}L_{\widehat{\gamma},11}^2+2L_{\widehat{\gamma},12}^2+L_{\widehat{\gamma},22}^2 &= (L_{\widehat{\gamma},11}+L_{\widehat{\gamma},22})^2-2(L_{\widehat{\gamma},11}L_{\widehat{\gamma},22}-L_{\widehat{\gamma},12}^2)\\&= 4H_{\widehat{\gamma}}^2W^2-2K_{\widehat{\gamma}}W^2.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\delta_{N_{\widehat{\gamma}}}^2W &= |\nabla\varphi|^2+2K_{\widehat{\gamma}}W\varphi^2\\&\quad+\sum_{\sigma=1}^{n}\left\{\left(\widehat{\gamma}_u^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,1}^\sigma\right)^2+\left(\widehat{\gamma}_v^\sigma+\sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right)^2\right\}\varphi^2\end{aligned}$$

for arbitrary $\varphi\in C_0^\infty(B,\mathbb{R})$. The statement follows. □

Invariant formulation

From the previous proof we can infer the form of the second variation using arbitrary parameters $(u,v) \in B$. For this purpose we need the following definition (see e.g. Blaschke and Leichtweiss [15]).

Definition 13.1. Let two functions $\phi, \psi \in C^1(B,\mathbb{R})$ be given. Its *Beltrami operator of first kind w.r.t. the metric*

$$ds^2 = g_{11}\,du^2 + 2g_{12}\,du\,dv + g_{22}\,dv^2$$

is defined as

$$\nabla_{ds^2}(\phi,\psi) := \sum_{i,j=1}^{2} g^{ij}\phi_{u^i}\psi_{u^j} \tag{13.1}$$

In particular, if $(u,v) \in B$ are conformal parameters, i.e. $g_{11} = W = g_{22}$ and $g_{12} = 0$, then there hold

$$\nabla_{ds^2}(\phi,\psi) = \frac{1}{W}(\phi_u\psi_u + \phi_v\psi_v) = \frac{1}{W}\nabla\phi\cdot\nabla\psi$$

with the Euclidean gradient operator $\nabla\phi = (\phi_u,\phi_v)$ etc., as well as

$$\nabla_{ds^2}(\phi,\phi) = \frac{1}{W}|\nabla\phi|^2.$$

The Beltrami operator of first kind is well defined for arbitrary Riemannian metrics $d\sigma^2$ with positive determinant.

Corollary 13.1. *Using an arbitrary parameter system $(u,v) \in B$ it holds*

$$\begin{aligned}\delta^2_{N_{\widehat{\gamma}}}\mathscr{A}[X;\varphi] &= \iint_B \nabla_{ds^2}(\varphi,\varphi)W\,du\,dv + 2\iint_B K_{\widehat{\gamma}}W\varphi^2\,du\,dv \\ &\quad + \sum_{i,j=1}^{2}\sum_{\sigma=1}^{n}\iint_B g^{ij}\left(\widehat{\gamma}_{u^i} + \sum_{\omega=1}^{n}\widehat{\gamma}^{\omega}T^{\sigma}_{\omega,i}\right)\left(\widehat{\gamma}_{u^j} + \sum_{\omega=1}^{n}\widehat{\gamma}^{\omega}T^{\sigma}_{\omega,j}\right)W\varphi^2\,du\,dv.\end{aligned}$$

Special cases

We want to exemplify the previous considerations.

Consider first the case $n = 1$ of one codimension. Up to orientation there is only one unit normal vector N for the minimal immersion $X\colon B \to \mathbb{R}^3$, and the formula for the second variation reduces to

$$\delta_N\mathscr{A}[X;\varphi] = \iint_B \left(|\nabla\varphi|^2 + 2KW\varphi^2\right)du\,dv.$$

Furthermore, instead of $\widetilde{X} = X + \varepsilon\varphi N_{\widehat{\gamma}}$ we will often consider special normal variations of the form

$$\widetilde{X}(u,v) = X(u,v) + \varepsilon\varphi(u,v)N_\omega(u,v).$$

Here the unit normal vector N_ω is chosen from a given ONF $N = (N_1,\ldots,N_n)$, i.e. we set

$$\widehat{\gamma}^\vartheta = \begin{cases} 1 & \text{if } \vartheta = \omega \\ 0 & \text{if } \vartheta \neq \omega \end{cases}.$$

Then the second variations of the area functional reads

$$\begin{aligned} \delta^2_{N_\omega}\mathscr{A}[X;\varphi] = &\iint\limits_B \left(|\nabla\varphi|^2 + 2K_\omega W\varphi^2\right) dudv \\ &+ \sum_{\sigma=1}^n \iint\limits_B \left\{(T^\sigma_{\omega,1})^2 + (T^\sigma_{\omega,2})^2\right\} \varphi^2\, dudv \end{aligned} \tag{13.2}$$

using conformal parameters.

The functional of the total torsion which appears here on the right hand side in combination with a test function $\varphi \in C_0^\infty(B,\mathbb{R})$ can be controlled by means of our estimates established in chapters 7 and 8 if we use normal Coulomb frames N of X for the construction of $\widetilde{X}$.

13.2 Immersions with a special mean curvature field

Next we want to compute the second variation of the functional

$$\mathscr{J}[X] := \iint\limits_B \Gamma(X)W\, dudv$$

which emerges from the general functional of Gulliver-type

$$\mathscr{G}[X] = \iint\limits_B \left\{\Gamma(X)W + X_u \circ \mathbf{A}(X) \circ X_v\right\} dudv$$

in the special case $\mathbf{A}(X) \equiv \mathbf{0}$.

From section 10.1 we first infer that critical points of $\mathscr{J}[X]$ possess the mean curvature field

$$H_\sigma(X,N_\sigma) = \frac{\Gamma_X(X)\cdot N_\sigma}{2\Gamma(X)}, \quad \sigma = 1,\ldots,n.$$

The case $\Gamma(X) \equiv 1$ corresponds to the area functional with minimal surfaces as critical points.

The second variation

Theorem 13.2. *Using conformal parameters* $(u,v) \in B$*, the second variation of* $\mathscr{J}[X]$ *w.r.t. to perturbations*

$$\widetilde{X}(u,v) = X(u,v) + \varepsilon\varphi(u,v)N_{\widehat{\gamma}}(u,v),$$

where

$$N_{\widehat{\gamma}}(u,v) = \sum_{\sigma=1}^{n} \widehat{\gamma}^{\sigma}(u,v)N_{\sigma}(u,v), \quad \sum_{\sigma=1}^{n} \widehat{\gamma}(u,v)^2 = 1,$$

$\widehat{\gamma} = (\widehat{\gamma}^1, \ldots, \widehat{\gamma}^n)$, $\varphi \in C_0^\infty(B,\mathbb{R})$ *and real* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *is given by*

$$\begin{aligned} \delta^2_{N_{\widehat{\gamma}}} \mathscr{J}[X;\varphi] = & \iint_B \Gamma(X)|\nabla\varphi|^2\, dudv \\ & + 2\iint_B \Big\{ H_{\widehat{\gamma},X}(X,N_{\widehat{\gamma}}) \cdot N_{\widehat{\gamma}} - 2H_{\widehat{\gamma}}(X,N_{\widehat{\gamma}})^2 + K_{\widehat{\gamma}} \Big\} \Gamma(X)W\varphi^2\, dudv \\ & + \sum_{\sigma=1}^{n} \iint_B \Big\{ \Big(\widehat{\gamma}^{\sigma}_u + \widehat{\gamma}^{\omega} T^{\sigma}_{\omega,1} \Big)^2 + \Big(\widehat{\gamma}^{\sigma}_v + \widehat{\gamma}^{\omega} T^{\sigma}_{\omega,2} \Big)^2 \Big\} \Gamma(X)\varphi^2\, dudv \end{aligned}$$

with the mean curvature $H_{\widehat{\gamma}} = H_{\widehat{\gamma}}(X,N_{\widehat{\gamma}})$ *and the Gauss curvature* $K_{\widehat{\gamma}}$ *of the immersion along* $N_{\widehat{\gamma}}$*. In particular, if the ONF N is free of torsion then the third integrand on the right hand side vanishes identically.*

Proof. First we compute

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\Gamma(\widetilde{X})\widetilde{W} &= \Gamma_X(\widetilde{X}) \cdot N_{\widehat{\gamma}}\widetilde{W}\varphi + \Gamma(\widetilde{X})\frac{\partial}{\partial\varepsilon}\widetilde{W} \\ &= 2\Gamma(\widetilde{X})H^*_{\widehat{\gamma}}(\widetilde{X})\widetilde{W}\varphi + \Gamma(\widetilde{X})\frac{\partial}{\partial\varepsilon}\widetilde{W} \end{aligned}$$

setting

$$H^*_{\sigma}(X) = \frac{\Gamma_X(X) \cdot N_{\sigma}}{2\Gamma(X)}$$

with N_σ fixed. Note here that in the first identity no variation $\widetilde{N}_{\widehat{\gamma}}$ appears. Thus we obtain

$$\frac{\partial}{\partial\varepsilon}\mathscr{J}[\widetilde{X}] = \iint_B \Big\{ 2\Gamma(\widetilde{X})H^*_{\widehat{\gamma}}(\widetilde{X})\widetilde{W}\varphi + \Gamma(\widetilde{X})\frac{\partial}{\partial\varepsilon}\widetilde{W} \Big\}\, dudv.$$

A further differentiation w.r.t. ε at $\varepsilon = 0$ taking account of the identities

$$\delta_{N_{\widehat{\gamma}}}W = -2H_{\widehat{\gamma}}(X,N_{\widehat{\gamma}})W\varphi = -2H^*_{\widehat{\gamma}}(X)W\varphi$$

as well as

$$\delta_{N_{\widehat{\gamma}}}^2 W = |\nabla\varphi|^2 + 2K_{\widehat{\gamma}} W\varphi^2 + \sum_{\sigma=1}^{n}\left\{\left(\widehat{\gamma}_u^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,1}^\sigma\right)^2 + \left(\widehat{\gamma}_v^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right)^2\right\}\varphi^2$$

from above yields

$$\begin{aligned}\delta_{N_{\widehat{\gamma}}}\mathscr{I}[X;\varphi] &= 2\iint_B \left\{\Gamma_X(X)\cdot N_{\widehat{\gamma}} H_{\widehat{\gamma}}^*(X) + \Gamma(X) H_{\widehat{\gamma},X}^*(X)\cdot N_{\widehat{\gamma}}\right\} W\varphi^2\,dudv \\ &\quad - 2\iint_B \left\{2\Gamma(X)H_{\widehat{\gamma}}^*(X)^2 + \Gamma_X(X)\cdot N_{\widehat{\gamma}} H_{\widehat{\gamma}}^*(X)\right\} W\varphi^2\,dudv \\ &\quad + \iint_B \Gamma(X)|\nabla\varphi|^2\,dudv + 2\iint_B \Gamma(X)K_{\widehat{\gamma}} W\varphi^2\,dudv \\ &\quad + \iint_B \sum_{\sigma=1}^{n}\left\{\left(\widehat{\gamma}_u^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,1}^\sigma\right)^2 + \left(\widehat{\gamma}_v^\sigma + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,2}^\sigma\right)^2\right\}\Gamma(X)\varphi^2\,dudv.\end{aligned}$$

Due to $H_{\widehat{\gamma},X}^*(X) = H_{\widehat{\gamma},X}(X,N_{\widehat{\gamma}})$, rearranging proves the statement. □

Invariant formulation

Finally we want to reformulate the result of the previous theorem in terms of arbitrary parametrizations. For this purpose we use the invariant Beltrami operator $\nabla_{ds^2}(\cdot,\cdot)$ from the foregoing section.

Corollary 13.2. *Using an arbitrary parameter system $(u,v)\in B$ it holds*

$$\begin{aligned}\delta_{N_{\widehat{\gamma}}}^2\mathscr{I}[X;\varphi] &= \iint_B \nabla_{ds^2}(\varphi,\varphi)\Gamma(X)W\,dudv \\ &\quad + \iint_B \left\{H_{\widehat{\gamma},X}(X,N_{\widehat{\gamma}})\cdot N_{\widehat{\gamma}} - 2H_{\widehat{\gamma}}(X,N_{\widehat{\gamma}})^2 + K_{\widehat{\gamma}}\right\}\Gamma(X)W\varphi^2\,dudv \\ &\quad + \sum_{i,j=1}^{2}\sum_{\sigma=1}^{n}\iint_B g^{ij}\left(\widehat{\gamma}_{u^i} + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,i}^\sigma\right)\left(\widehat{\gamma}_{u^j} + \sum_{\omega=1}^{n}\widehat{\gamma}^\omega T_{\omega,j}^\sigma\right)\Gamma(X)W\varphi^2\,dudv\end{aligned}$$

for all $\varphi\in C_0^\infty(B,\mathbb{R})$.

13.3 Stability and μ-stability

The most general geometric functional we have been considering so far is the following functional of Gulliver-type

$$\mathscr{G}[X] = \iint_B \left\{ \Gamma(X)W + 2X_u \circ \mathbf{A}(X) \circ X_v \right\} dudv$$

whose critical points possess the mean curvature field

$$H_\sigma(X,N_\sigma) = \frac{1}{\Gamma(X)W} \sum_{k,\ell,m=1}^{n+2} a_{k\ell,x^m}(x_u^k x_v^\ell n_\sigma^m + x_u^m x_v^k n_\sigma^\ell + x_u^\ell x_v^m n_\sigma^k) + \frac{\Gamma_X(X)\cdot N_\sigma}{2\Gamma(X)}\,.$$

Non-negativity of the second variation of $\mathscr{G}[X]$ for critical points X leads us to the concept of *(weakly) stable immersions.* We will consider stable minimal surfaces and stable immersions with a special prescribed mean curvature $H(X)$.

More general is the concept of *μ-stability.* In this section we will verify a certain μ-stability condition for stable immersions. In the sections to follow as well as in the next chapter we will establish μ-stability conditions *without assuming stability of the immersions under consideration.*

Stable immersions

Definition 13.2. Let $X\colon B \longrightarrow \mathbb{R}^{n+2}$ be critical for the functional $\mathscr{G}[X]$. We say that *the immersion is additionally stable for* $\mathscr{G}[X]$ if it holds

$$\delta^2_{N_{\widehat{\gamma}}}\mathscr{G}[X;\varphi] \ge 0$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$ and for all unit normal vectors $N_{\widehat{\gamma}}$.

Examples

Let us come back to formula (13.2) for a minimal immersion $X\colon B \longrightarrow \mathbb{R}^{n+2}$ with ONF N. If the surface is stable then

$$\iint_B |\nabla\varphi|^2\,dudv \ge 2\iint_B (-K_\omega)W\varphi^2\,dudv - \sum_{\sigma=1}^{n} \iint_B \left\{ (T_{\omega,1}^\sigma)^2 + (T_{\omega,2}^\sigma)^2 \right\} \varphi^2\,dudv$$

for all $\omega = 1,\ldots,n$.

Notice that $-K_\omega \geq 0$. Summation implies

$$\iint\limits_B |\nabla\varphi|^2\,dudv \geq \frac{2}{n}\iint\limits_B (-K)W\varphi^2\,dudv - \frac{1}{n}\sum_{\sigma,\omega=1}^{n}\iint\limits_B \Big\{(T_{\omega,1}^{\sigma})^2 + (T_{\omega,2}^{\sigma})^2\Big\}\,\varphi^2\,dudv$$

with the Gaussian curvature

$$K = \sum_{\omega=1}^{n} K_\omega\,.$$

Consider now the second variation formula for the functional $\mathscr{J}[X]$. Evaluating it for N_ω from the ONF N instead of the general field $N_{\widehat{\gamma}}$ gives us

$$\begin{aligned}\iint\limits_B \Gamma(X)|\nabla\varphi|^2\,dudv \geq{}& 2\iint\limits_B \Big\{2H_\omega(X,N_\omega)^2 - K_\omega\Big\}\Gamma(X)W\varphi^2\,dudv\\ &- 2\iint\limits_B H_{\omega,X}(X,N_\omega)\cdot N_\omega\Gamma(X)W\varphi^2\,dudv\\ &- \sum_{\sigma=1}^{n}\iint\limits_B \Big\{(T_{\omega,1}^{\sigma})^2 + (T_{\omega,2}^{\sigma})^2\Big\}\Gamma(X)W\varphi^2\,dudv.\end{aligned}$$

Summation over $\omega = 1,\ldots,n$ shows now

$$\begin{aligned}\iint\limits_B \Gamma(X)|\nabla\varphi|^2\,dudv \geq{}& \frac{2}{n}\iint\limits_B \Big\{2H(X)^2 - K\Big\}\Gamma(X)W\varphi^2\,dudv\\ &- \frac{2}{n}\sum_{\omega=1}^{n}\iint\limits_B H_{\omega,X}(X,N_\omega)\cdot N_\omega\Gamma(X)W\varphi^2\,dudv\\ &- \frac{1}{n}\sum_{\sigma,\omega=1}^{n}\iint\limits_B \Big\{(T_{\omega,1}^{\sigma})^2 + (T_{\omega,2}^{\sigma})^2\Big\}\Gamma(X)W\varphi^2\,dudv\end{aligned}$$

with the square $H(X)^2$ of the mean curvature vector

$$H(X) = \sum_{\omega=1}^{n} H_\omega(X,N_\omega)N_\omega\,.$$

μ-stable immersions

Motivated from these examples we introduce the following general concept.

Definition 13.3. The immersion X is called *μ-stable with a real number $\mu > 0$* if there is a smooth function $q \in C^{2+\alpha}(B,\mathbb{R})$ such that it holds

$$\begin{aligned}\iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,dudv \geq \mu \iint\limits_B (q-K)W\varphi^2\,dudv \\ -\alpha(n)\sum_{i,j=1}^{2}\sum_{\sigma,\vartheta=1}^{n}\iint\limits_B g^{ij}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}W\varphi^2\,dudv\end{aligned} \tag{13.3}$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$, where $\alpha(n)$ is a non-negative real number depending on the codimension $n \geq 1$.

Generalized stability conditions were studied in the literature. We especially want to refer the reader to Fischer-Colbrie and Schoen [60], Colding and Minicizzi [36] and Fröhlich [62].

Examples

Let us first examplarily consider the stability inequality

$$\begin{aligned}\iint\limits_B |\nabla\varphi|^2\,dudv \geq \frac{2}{n}\iint\limits_B (-K)W\varphi^2\,dudv \\ -\frac{1}{n}\sum_{\sigma,\omega=1}^{n}\iint\limits_B \Big\{(T_{\omega,1}^{\sigma})^2+(T_{\omega,2}^{\sigma})^2\Big\}W\varphi^2\,dudv\end{aligned}$$

for a conformally parametrized minimal surface $X\colon B \to \mathbb{R}^{n+2}$. Obviously this surface is μ-stable with

$$\mu = \frac{2}{n} \quad \text{and} \quad \alpha(n) = \frac{1}{n}, \quad q \equiv 0.$$

The case $n = 1$ of one codimension leads us to μ-stable minimal surfaces with $\mu = 2$, i.e.

$$\iint\limits_B |\nabla\varphi|^2\,dudv \geq 2\iint\limits_B (-K)W\varphi^2\,dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$. In this situation the usual stability inequality coincides with the μ-stability with $\mu = 2$.

Secondly we consider the stability inequality we previously derived for $\mathscr{J}$-critical immersions $X\colon B \to \mathbb{R}^{n+2}$. Let again two positive real numbers

$$0 < \Gamma_0 \leq \Gamma(X) \leq \Gamma_1 < \infty \quad \text{for all } X \in \mathbb{R}^{n+2}$$

be given.

Assume that for all $X \in \mathbb{R}^{n+2}$ the following smallness condition is satisfied

$$|H_{\omega,X}(X,N_\omega)| \le H_\omega(X,N_\omega)^2 \quad \text{for all } \omega = 1,\ldots,n.$$

Then we conclude

$$\iint\limits_B |\nabla\varphi|^2\,dudv \ge \frac{2\Gamma_0}{n\Gamma_1}\iint\limits_B \left\{\left[2H(X)^2 - K\right] - \sum_{\omega=1}^n H_{\omega,X}(X,N_\omega)\cdot N_\omega\right\} W\varphi^2\,dudv$$
$$-\frac{1}{n}\sum_{\sigma,\omega=1}^n \iint\limits_B \left\{(T_{\omega,1}^\sigma)^2 + (T_{\omega,2}^\sigma)^2\right\} W\varphi^2\,dudv$$

where we naturally assume the right hand side be non-negative for all $\varphi \in C_0^\infty(B,\mathbb{R})$.

The following sections are devoted to the problem of establishing μ-stability conditions from various geometric conditions. Further considerations regarding weighted minimal surfaces can be found in the next chapter.

13.4 μ-stability due to Schwarz for minimal graphs

A classical result due to H.A. Schwarz is (see e.g. Nitsche [126], §104).

Given the conformally parametrized minimal immersion X in $\mathbb{R}^3$, and assume that there is an everywhere positive solution χ of the differential equation

$$\triangle\chi - 2KW\chi = 0 \quad \text{in } B.$$

Then X is stable in the sense that

$$\iint\limits_B |\nabla\varphi|^2\,dudv \ge 2\iint\limits_B (-K)W\varphi^2\,dudv \quad \text{for all } \varphi \in C_0^\infty(B,\mathbb{R}).$$

Now we will prove this result in the general case of higher codimension *but for surfaces with flat normal bundle.*

An auxiliary function

For this purpose we introduce the function

$$\chi := \frac{x_u^1 x_v^2 - x_v^1 x_u^2}{W} \quad \text{with the Jacobian } J_F := x_u^1 x_v^2 - x_u^2 x_v^1 \qquad (13.4)$$

of the plane mapping $F = (x^1,x^2)$ for the surface vector $X = (x^1,x^2,\ldots,x^{n+2})$.

It holds $\chi > 0$ if X represents a graph over the $[x,y]$-plane. Furthermore using conformal parameters $(u,v) \in B$ we have

$$\Delta\chi - 2KW\chi = 0 \quad \text{in } B$$

for the minimal graph if its normal bundle is flat.

The general case is contained in our next

Proposition 13.1. *Let $X\colon B \to \mathbb{R}^{n+2}$ be a conformally parametrized minimal surface with an ONF N be given. Then it holds*

$$\Delta\chi = 2KW\chi + \sum_{\sigma,\vartheta=1}^{n} S_{\sigma,12}^{\vartheta}(n_\sigma^1 n_\vartheta^2 - n_\sigma^2 n_\vartheta^1) \quad \textit{in } B$$

with the Gaussian curvature K of the surface and the components $S_{\sigma,ij}^{\vartheta}$ of the curvature tensor of its normal bundle.

If $n = 1$ then χ is represents the third component of the unit normal vector N of the immersion $X\colon B \to \mathbb{R}^3$. In fact it holds

$$\Delta N = 2KWN \quad \text{in } B$$

using conformal parameters for minimal surfaces.

An elliptic system for the vector $\mathscr{N}$

The differential equation from the previous proposition follows at once from an elliptic system for the unit vector

$$\mathscr{N} = \frac{X_u \wedge X_v}{W}$$

which we introduced in chapter 6.

Theorem 13.3. *Let the conformally parametrized minimal surface $X\colon B \to \mathbb{R}^{n+2}$ be given. Then it holds*

$$\Delta\mathscr{N} = 2KW\mathscr{N} + \sum_{\sigma,\vartheta=1}^{n} S_{\sigma,12}^{\vartheta} N_\sigma \wedge N_\vartheta = 2KW\mathscr{N} + 2\mathscr{S}W$$

with the Grassmann curvature vector

$$\mathscr{S} = \frac{1}{W} \sum_{1\le\sigma<\vartheta\le n} S_{\sigma,12}^{\vartheta} N_\sigma \wedge \vartheta$$

of the normal bundle from chapter 6.

Proof. Let us consider the function

$$\widetilde{\chi} := \frac{x_u^k x_v^\ell - x_u^\ell x_v^k}{W}, \quad \widetilde{F} := x_u^k x_v^\ell - x_u^\ell x_v^k, \quad k,\ell = 1,\ldots,n+2.$$

We will prove the identity

$$\Delta\widetilde{\chi} = 2KW\widetilde{\chi} + \sum_{\sigma,\vartheta=1}^{n} S_{\sigma,12}^{\vartheta}(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k).$$

We start with computing the first and second derivatives of $\widetilde{\chi}$. First we have

$$\begin{aligned}
\widetilde{\chi}_u &= \frac{1}{W}\sum_{\sigma=1}^{n}\left\{L_{\sigma,11}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\right\},\\
\widetilde{\chi}_v &= \frac{1}{W}\sum_{\sigma=1}^{n}\left\{L_{\sigma,12}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,22}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\right\}.
\end{aligned}$$

For the proof we calculate

$$\begin{aligned}
\widetilde{\chi}_u &= \frac{J_{\widetilde{F},u}}{W} - \frac{W_u}{W^2}J_{\widetilde{F}} = \frac{1}{W}(x_{uu}^k x_v^\ell + x_u^k x_{uv}^\ell - x_{uu}^\ell x_v^k - x_u^\ell x_{uv}^k) - \frac{W_u}{W^2}J_{\widetilde{F}}\\
&= \frac{1}{W}\Big\{\Gamma_{11}^1 x_u^k x_v^\ell + \Gamma_{11}^2 x_v^k x_v^\ell + \Gamma_{12}^1 x_u^\ell x_u^k + \Gamma_{12}^2 x_v^\ell x_u^k - \Gamma_{11}^1 x_u^\ell x_v^k - \Gamma_{11}^2 x_v^\ell x_v^k - \ldots\\
&\qquad \ldots - \Gamma_{12}^1 x_u^k x_u^\ell - \Gamma_{12}^2 x_v^k x_u^\ell\Big\} - \frac{W_u}{W^2}J_{\widetilde{F}}\\
&\quad + \frac{1}{W}\sum_{\sigma=1}^{n}(L_{\sigma,11}n_\sigma^k x_v^\ell + L_{\sigma,12}n_\sigma^\ell x_u^k - L_{\sigma,11}n_\sigma^\ell x_v^k - L_{\sigma,12}n_\sigma^\ell x_u^k)\\
&= \frac{1}{2W^2}\Big\{W_u x_u^k x_v^\ell - W_v x_v^k x_v^\ell + W_v x_u^\ell x_u^k + W_u x_v^\ell x_u^k - W_u x_u^\ell x_v^k + W_v x_v^\ell x_v^k - \ldots\\
&\qquad \ldots - W_v x_u^k x_u^\ell - W_u x_v^k x_v^\ell\Big\} - \frac{W_u}{W^2}J_{\widetilde{F}}\\
&\quad + \frac{1}{W}\sum_{\sigma=1}^{n}\left\{L_{\sigma,11}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\right\}\\
&= \frac{1}{2W^2}\Big\{W_u x_u^k x_v^\ell + W_u x_v^\ell x_u^k - W_u x_u^\ell x_v^k - W_u x_v^k x_u^\ell\Big\} - \frac{W_u}{W^2}J_{\widetilde{F}}\\
&\quad + \frac{1}{W}\sum_{\sigma=1}^{n}\frac{L_{\sigma,11}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)}{W}\\
&= \frac{1}{W}\sum_{\sigma=1}^{n}\left\{L_{\sigma,11}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\right\}.
\end{aligned}$$

Analogously we continue with the derivative w.r.t. v. There follow

$$
\begin{aligned}
\widetilde{\chi}_{uu} = &-\frac{W_u}{W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,12}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,22}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,11,u}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12,u}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,11}(n_{\sigma,u}^k x_v^\ell - n_{\sigma,u}^\ell x_v^k) - L_{\sigma,12}(n_{\sigma,u}^k x_u^\ell - n_{\sigma,u}^\ell x_u^k)\\
&+\frac{1}{W}\sum_{\sigma=1}^{n} L_{\sigma,11}(n_\sigma^k x_{uv}^\ell - n_\sigma^\ell x_{uv}^k) - L_{\sigma,12}(n_\sigma^k x_{uu}^\ell - n_\sigma^\ell x_{uu}^k)\\
= &-\frac{W_u}{W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,12}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,22}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,11,u}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12,u}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&-\frac{1}{W^2}\sum_{\sigma=1}^{n} L_{\sigma,11}(L_{\sigma,11}x_u^k x_v^\ell + L_{\sigma,12}x_v^k x_v^\ell - L_{\sigma,11}x_u^\ell x_v^k - L_{\sigma,12}x_v^\ell x_v^k)\\
&+L_{\sigma,12}(L_{\sigma,11}x_u^k x_u^\ell + L_{\sigma,12}x_v^k x_u^\ell - L_{\sigma,11}x_u^\ell x_u^k - L_{\sigma,12}x_v^\ell x_v^k)\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}\Big\{L_{\sigma,11}T_{\sigma,1}^{\vartheta}(n_\vartheta^k x_v^\ell - n_\vartheta^\ell x_v^k) - L_{\sigma,12}T_{\sigma,1}^{\vartheta}(n_\vartheta^k x_u^\ell - n_\vartheta^\ell x_u^k)\Big\}\\
&+\frac{1}{W^2}\sum_{\sigma=1}^{n} L_{\sigma,11}(\Gamma_{12}^1 x_u^\ell n_\sigma^k + \Gamma_{12}^2 x_v^\ell n_\sigma^k - \Gamma_{12}^1 x_u^k n_\sigma^\ell - \Gamma_{12}^2 x_v^k n_\sigma^\ell)\\
&-L_{\sigma,12}(\Gamma_{11}^1 x_u^\ell n_\sigma^k + \Gamma_{11}^2 x_v^\ell n_\sigma^k - \Gamma_{12}^1 x_u^k n_\sigma^\ell - \Gamma_{12}^2 x_v^k n_\sigma^\ell)\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}\Big\{L_{\sigma,11}L_{\vartheta,12}(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k) - L_{\sigma,12}L_{\vartheta,11}(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{W_u}{2W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,11}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,12}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&\quad -\frac{1}{W}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,22,u}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) + L_{\sigma,12,u}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&\quad +\frac{1}{2W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,11}W_v(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k) + L_{\sigma,11}W_u(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)\Big\}\\
&\quad -\frac{1}{2W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,12}W_u(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k) - L_{\sigma,12}W_v(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)\Big\}\\
&\quad +\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,11}L_{\vartheta,12} - L_{\sigma,12}L_{\vartheta,11})(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k)\\
&\quad -\frac{1}{W^2}\sum_{\sigma=1}^{n}(L_{\sigma,11}^2 + L_{\sigma,12}^2)(x_u^k x_v^\ell - x_v^k x_u^\ell)\\
&\quad +\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}L_{\sigma,11}T_{\sigma,1}^{\vartheta}(n_\vartheta^k x_v^\ell - n_\vartheta^\ell x_v^k)\\
&\quad -\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}L_{\sigma,12}T_{\sigma,1}^{\vartheta}(n_\vartheta^k x_u^\ell - n_\vartheta^\ell x u^k)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\widetilde{\chi}_{vv} &= -\frac{1}{W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,12}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) - L_{\sigma,22}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&\quad +\frac{1}{W}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,12,v}(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) + L_{\sigma,11,v}(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}\\
&\quad -\frac{1}{2W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,12}W_u(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k) - L_{\sigma,12}W_v(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)\Big\}\\
&\quad -\frac{1}{2W^2}\sum_{\sigma=1}^{n}\Big\{L_{\sigma,22}W_u(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k) + L_{\sigma,22}W_v(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\Big\}
\end{aligned}
$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,12}L_{\vartheta,22}-L_{\sigma,22}L_{\vartheta,12})(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k)$$

$$-\frac{1}{W^2}\sum_{\sigma=1}^{n}(L_{\sigma,12}^2+L_{\sigma,22}^2)(x_u^k x_v^\ell - x_u^\ell x_v^k)$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}L_{\sigma,12}T_{\sigma,2}^{\vartheta}(n_\vartheta^k x_v^\ell - n_\vartheta^\ell x_v^k)-\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}L_{\sigma,22}T_{\sigma,2}^{\vartheta}(n_\vartheta^k x_u^\ell - n_\vartheta^\ell x_u^k).$$

Now the Laplacian:

$$\triangle\widetilde{\chi} = -\frac{1}{W^2}\sum_{\sigma=1}^{n}\Big(L_{\sigma,11}W_u - \frac{1}{2}L_{\sigma,11}W_u - \frac{1}{2}L_{\sigma,12}W_v + L_{\sigma,12}W_v - \dots$$

$$\dots - \frac{1}{2}L_{\sigma,12}W_v + \frac{1}{2}L_{\sigma,22}W_u\Big)(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)$$

$$-\frac{1}{W^2}\sum_{\sigma=1}^{n}\Big(-L_{\sigma,12}W_u - \frac{1}{2}L_{\sigma,11}W_v + \frac{1}{2}L_{\sigma,12}W_u - L_{\sigma,22}W_v + \dots$$

$$\dots + \frac{1}{2}L_{\sigma,12}W_u + \frac{1}{2}L_{\sigma,22}W_v\Big)(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}(-L_{\sigma,22,u}+L_{\sigma,12,v})(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}(-L_{\sigma,12,u}+L_{\sigma,11,v})(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)$$

$$-\frac{1}{W^2}\sum_{\sigma=1}^{n}(L_{\sigma,11}^2+2L_{\sigma,12}^2+L_{\sigma,22}^2)(x_u^k x_v^\ell - x_u^\ell x_v^k)$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,12}L_{\vartheta,22}-L_{\sigma,22}L_{\vartheta,12}+L_{\sigma,11}L_{\vartheta,12}-\dots$$

$$\dots - L_{\sigma,12}L_{\vartheta,11})(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k)$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,12}T_{\sigma,2}^{\vartheta}+L_{\sigma,11}T_{\sigma,1}^{\vartheta})(n_\vartheta^k x_v^\ell - n_\vartheta^\ell x_v^k)$$

$$-\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,22}T_{\sigma,2}^{\vartheta}+L_{\sigma,12}T_{\sigma,1}^{\vartheta})(n_{\vartheta}^{k}x_{u}^{\ell}-n_{\vartheta}^{\ell}x_{u}^{k})$$

$$=-\frac{1}{W^2}\sum_{\sigma=1}^{n}\Big(-L_{\sigma,12}W_u-\frac{1}{2}L_{\sigma,11}W_v+\frac{1}{2}L_{\sigma,12}W_u-L_{\sigma,22}W_v+\ldots$$

$$\ldots+\frac{1}{2}L_{\sigma,12}W_u+\frac{1}{2}L_{\sigma,22}W_v\Big)(n_{\sigma}^{k}x_{u}^{\ell}-n_{\sigma}^{\ell}x_{u}^{k})$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}(-L_{\sigma,22,u}+L_{\sigma,12,v})(n_{\sigma}^{k}x_{v}^{\ell}-n_{\sigma}^{\ell}x_{v}^{k})$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}(-L_{\sigma,12,u}+L_{\sigma,11,v})(n_{\sigma}^{k}x_{u}^{\ell}-n_{\sigma}^{\ell}x_{u}^{k})$$

$$-\frac{1}{W^2}\sum_{\sigma=1}^{n}(L_{\sigma,11}^2+2L_{\sigma,12}^2+L_{\sigma,22}^2)(x_{u}^{k}x_{v}^{\ell}-x_{u}^{\ell}x_{v}^{k})$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,12}L_{\vartheta,22}-L_{\sigma,22}L_{\vartheta,12}+L_{\sigma,11}L_{\vartheta,12}-\ldots$$

$$\ldots-L_{\sigma,12}L_{\vartheta,11})(n_{\sigma}^{k}n_{\vartheta}^{\ell}-n_{\sigma}^{\ell}n_{\vartheta}^{k})$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,12}T_{\sigma,2}^{\vartheta}+L_{\sigma,11}T_{\sigma,1}^{\vartheta})(n_{\vartheta}^{k}x_{v}^{\ell}-n_{\vartheta}^{\ell}x_{v}^{k})$$

$$-\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,22}T_{\sigma,2}^{\vartheta}+L_{\sigma,12}T_{\sigma,1}^{\vartheta})(n_{\vartheta}^{k}x_{u}^{\ell}-n_{\vartheta}^{\ell}x_{u}^{k})$$

$$=\frac{1}{W}\sum_{\sigma=1}^{n}(\Gamma_{22}^{1}L_{\sigma,11}+\Gamma_{22}^{2}L_{\sigma,12}-\Gamma_{12}^{1}L_{\sigma,12}-\Gamma_{12}^{2}L_{\sigma,22})(n_{\sigma}^{k}x_{v}^{\ell}-n_{\sigma}^{\ell}x_{v}^{k})$$

$$+\frac{1}{W}\sum_{\sigma=1}^{n}(\Gamma_{12}^{1}L_{\sigma,11}+\Gamma_{12}^{2}L_{\sigma,12}-\Gamma_{11}^{1}L_{\sigma,12}-\Gamma_{11}^{2}L_{\sigma,22})(n_{\sigma}^{k}x_{u}^{\ell}-n_{\sigma}^{\ell}x_{u}^{k})$$

$$-\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\vartheta,22}T_{\vartheta,1}^{\sigma}-L_{\vartheta,12}T_{\vartheta,2}^{\sigma})(n_{\sigma}^{k}x_{v}^{\ell}-n_{\sigma}^{\ell}x_{v}^{k})$$

$$- \frac{1}{W} \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} (L_{\vartheta,12} T^{\sigma}_{\vartheta,1} - L_{\vartheta,11} T^{\sigma}_{\vartheta,2})(n^k_\sigma x^\ell_u - n^\ell_\sigma x^k_u)$$

$$+ \frac{1}{W} \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} \Big\{ (L_{\sigma,11} - L_{\sigma,22}) L_{\vartheta,12} - (L_{\vartheta,11} - L_{\vartheta,22}) L_{\sigma,12} \Big\} (n^k_\sigma n^\ell_\vartheta - n^\ell_\sigma n^k_\vartheta)$$

$$- \frac{1}{W^2} \sum_{\sigma=1}^{n} (L^2_{\sigma,11} + 2L^2_{\sigma,12} + L^2_{\sigma,22})(x^k_u x^\ell_v - x^\ell_u x^k_v)$$

$$= \frac{1}{2W^2} \sum_{\sigma=1}^{n} (-W_u L_{\sigma,11} + W_v L_{\sigma,12} - W_v L_{\sigma,12} - W_u L_{\sigma,22})(n^k_\sigma x^\ell_v - n^\ell_\sigma x^k_v)$$

$$+ \frac{1}{2W^2} \sum_{\sigma=1}^{n} (W_v L_{\sigma,11} + W_u L_{\sigma,12} - W_u L_{\sigma,12} + W_v L_{\sigma,22})(n^k_\sigma x^\ell_u - n^\ell_\sigma x^k_u)$$

$$+ \frac{1}{W} \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} (L_{\vartheta,22} T^{\sigma}_{\vartheta,1} - L_{\vartheta,12} T^{\sigma}_{\vartheta,2})(n^k_\sigma x^\ell_v - n^\ell_\sigma x^k_v)$$

$$+ \frac{1}{W} \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} (L_{\vartheta,12} T^{\sigma}_{\vartheta,1} - L_{\vartheta,11} T^{\sigma}_{\vartheta,2})(n^k_\sigma x^\ell_u - n^\ell_\sigma x^k_u)$$

$$+ \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} S^{\vartheta}_{\sigma,12} (n^k_\sigma n^\ell_\vartheta - n^\ell_\sigma n^k_\vartheta)$$

$$- \frac{1}{W^2} \sum_{\sigma=1}^{n} (L^2_{\sigma,11} + 2L^2_{\sigma,12} + L^2_{\sigma,22})(x^k_u x^\ell_v - x^\ell_u x^k_v)$$

$$= \frac{1}{W} \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} (L_{\vartheta,22} T^{\sigma}_{\vartheta,1} - L_{\vartheta,12} T^{\sigma}_{\vartheta,2})(n^k_\sigma x^\ell_v - n^\ell_\sigma x^k_v)$$

$$+ \frac{1}{W} \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} (L_{\vartheta,12} T^{\sigma}_{\vartheta,1} - L_{\vartheta,11} T^{\sigma}_{\vartheta,2})(n^k_\sigma x^\ell_u - n^\ell_\sigma x^k_u$$

$$+ \sum_{\sigma=1}^{n} \sum_{\vartheta=1}^{n} S^{\vartheta}_{\sigma,12} (n^k_\sigma n^\ell_\vartheta - n^\ell_\sigma n^k_\vartheta)$$

$$
\begin{aligned}
&-2\sum_{\sigma=1}^{n}(-K_\varepsilon)(x_u^k x_v^\ell - x_u^\ell x_v^k)\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,11}T_{\sigma,1}^{\vartheta}+L_{\sigma,12}T_{\sigma,2}^{\vartheta})(n_\vartheta^k x_v^\ell - n_\vartheta^\ell x_v^k)\\
&-\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\sigma,22}T_{\sigma,2}^{\vartheta}+L_{\sigma,12}T_{\sigma,1}^{\vartheta})(n_\vartheta^k x_u^\ell - n_\vartheta^\ell x_u^k)\\
=&\ \frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\vartheta,22}T_{\vartheta,1}^{\sigma}-L_{\vartheta,12}T_{\vartheta,2}^{\sigma})(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\vartheta,12}T_{\vartheta,1}^{\sigma}-L_{\vartheta,11}T_{\vartheta,2}^{\sigma})(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\\
&+\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\vartheta,11}T_{\vartheta,1}^{\sigma}+L_{\vartheta,12}T_{\vartheta,2}^{\sigma})(n_\sigma^k x_v^\ell - n_\sigma^\ell x_v^k)\\
&-\frac{1}{W}\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}(L_{\vartheta,22}T_{\vartheta,2}^{\sigma}+L_{\vartheta,12}T_{\vartheta,1}^{\sigma})(n_\sigma^k x_u^\ell - n_\sigma^\ell x_u^k)\\
&+\sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n}S_{\sigma,12}^{\vartheta}(n_\sigma^k n_\vartheta^\ell - n_\sigma^\ell n_\vartheta^k)-2\sum_{\sigma=1}^{n}(-K_\sigma)W\widetilde{\chi}
\end{aligned}
$$

proving the statement. □

μ-stability for minimal graphs

We want to construct a real number $\mu > 0$ such that the following μ-stability inequality is true

$$\iint_B |\nabla\varphi|^2\,dudv \ge \mu \iint_B (-K)W\varphi^2\,dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$ using conformal parameters. For this purpose we recall the Ricci integrability condition

$$S_{\sigma,12}^{\vartheta} = \frac{1}{W}(L_{\sigma,11}-L_{\sigma,22})L_{\vartheta,12} - \frac{1}{W}(L_{\vartheta,11}-L_{\vartheta,22})L_{\sigma,12}$$

from chapter 3, formula (3.14).

We estimate as follows

$$
\begin{aligned}
|S^{\vartheta}_{\sigma,12}| &\le \frac{1}{4W}\,(L_{\sigma,11}-L_{\sigma,22})^2+\frac{1}{W}\,L^2_{\sigma,12}+\frac{1}{4W}\,(L_{\vartheta,11}-L_{\vartheta,22})^2+\frac{1}{W}\,L^2_{\vartheta,12}\\
&=\frac{1}{4W}\,(L^2_{\sigma,11}+2L_{\sigma,11}L_{\sigma,22}+L^2_{\sigma,22})-\frac{1}{W}\,(L_{\sigma,11}L_{\sigma,22}-L^2_{\sigma,12})\\
&\quad+\frac{1}{4W}\,(L^2_{\vartheta,11}+2L_{\vartheta,11}L_{\vartheta,22}+L^2_{\vartheta,22})-\frac{1}{W}\,(L_{\vartheta,11}L_{\vartheta,22}-L^2_{\vartheta,12})
\end{aligned}
$$

leading to

$$|S^{\vartheta}_{\sigma,12}|\le H^2_{\sigma}W-K_{\sigma}W+H^2_{\vartheta}W-K_{\vartheta}W=-K_{\sigma}W-K_{\vartheta}W$$

where the right hand side is non-negative due to $H_\sigma\equiv 0$ for all $\sigma=1,\ldots,n$.

Theorem 13.4. *Let $X\colon B\to\mathbb{R}^{n+2}$ be conformally reparametrization of minimal graph in $\mathbb{R}^{n+2}$ together with an ONF N. Assume furtermore*

$$0<\chi_0\le\chi=\frac{x^1_u x^2_v-x^2_u x^1_v}{W}$$

holds true with a real number $\chi_0>0$ as well as

$$|n^1_\sigma n^2_\vartheta-n^2_\sigma n^1_\vartheta|\le N_1<\infty\quad\textit{for all }\sigma,\vartheta=1,\ldots,n$$

with real $N_1\in[0,\infty)$ such that additionally

$$\chi_0<nN_1$$

is satisfied. Then the graph is stable in the sense of

$$\iint\limits_B|\nabla\varphi|\,dudv\ge\mu\iint\limits_B(-K)W\varphi^2\,dudv$$

for all $\varphi\in C^\infty_0(B,\mathbb{R})$ where the stability constant μ is chosen such that

$$0<\mu\le 2-\frac{2nN_1}{\chi_0}\,.$$

The following proof shows that we can replace the assumption $0<\chi_0\le\chi$ by $0<\chi$ if $n=1$. It is clear that in this case $N_1=0$.

Corollary 13.3. *A minimal graph $X\colon B\to\mathbb{R}^3$ is μ-stable with $\mu=2$, i.e. it holds*

$$\iint\limits_B|\nabla\varphi|^2\,dudv\ge 2\iint\limits_B(-K)W\varphi^2\,dudv$$

for all $\varphi\in C^\infty_0(B,\mathbb{R})$ using conformal parameters $(u,v)\in B$.

The same stability inequality holds true if the minimal graph $X\colon B \to \mathbb{R}^{n+2}$ has flat normal bundle.

Corollary 13.4. *A minimal graph $X\colon B \to \mathbb{R}^{n+2}$ with flat normal bundle is μ-stable with $\mu = 2$, i.e. it holds*

$$\iint_B |\nabla\varphi|^2\,dudv \ge 2\iint_B (-K)W\varphi^2\,dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$ using conformal parameters $(u,v)\in B$.

Both these results arise from the following proof of the Theorem.

Proof of the Theorem. We know that the Weingarten equations imply

$$\sum_{\sigma=1}^n |\nabla N_\sigma|^2 = 2\sum_{\sigma=1}^n (-K_\sigma)W = 2(-K)W$$

for the Gauss curvature K. Let $\varphi \in C_0^\infty(B,\mathbb{R})$ be given. We compute $|\nabla(\varphi\chi^{-1})|^2$ and arrive at

$$\begin{aligned}|\nabla\varphi|^2 &= \chi^2|\nabla(\varphi\chi^{-1})|^2 + \operatorname{div}\left(\frac{\varphi^2\chi_u}{\chi},\frac{\varphi^2\chi_v}{\chi}\right) - \frac{\varphi^2}{\chi}\triangle\chi \\ &= \chi^2|\nabla(\varphi\chi^{-1})|^2 + \operatorname{div}\left(\frac{\varphi^2\chi_u}{\chi},\frac{\varphi^2\chi_v}{\chi}\right) + 2(-K)W\varphi^2 \\ &\quad - \frac{1}{\chi}\sum_{\sigma,\vartheta=1}^n S_{\sigma,12}^\vartheta (n_\sigma^1 n_\vartheta^2 - n_\sigma^2 n_\vartheta^1)\varphi^2\end{aligned}$$

Partial integration gives

$$\begin{aligned}&\iint_B \left\{|\nabla\varphi|^2 - \mu(-K)W\varphi^2\right\}dudv \\ &\quad\ge \iint_B \chi^2|\nabla(\varphi\chi^{-1})|^2\,dudv + (2-\mu)\iint_B (-K)W\varphi^2\,dudv \\ &\qquad - \sum_{\sigma,\vartheta=1}^n \iint_B \frac{1}{\chi}|S_{\sigma,12}^\vartheta||n_\sigma^1 n_\vartheta^2 - n_\sigma^2 n_\vartheta^1|\varphi^2\,dudv \\ &\quad\ge (2-\mu)\iint_B (-K)W\varphi^2\,dudv - \frac{N_1}{\chi_0}\sum_{\sigma,\vartheta=1}^n \iint_B (-K_\sigma - K_\vartheta)W\varphi^2\,dudv \\ &\quad= (2-\mu)\iint_B (-K)W\varphi^2\,dudv - \frac{2nN_1}{\chi_0}\iint_B (-K)W\varphi^2\,dudv.\end{aligned}$$

Thus we are led to

$$\iint\limits_B \Big\{ |\nabla\varphi|^2 - \mu(-K)W\varphi^2 \Big\}\, dudv \ge \Big\{ 2 - \mu - \frac{2nN_1}{\chi_0} \Big\} \iint\limits_B (-K)W\varphi^2\, dudv$$

where the right hand side is non-negative due to our assumptions. □

We want to give a representation of $\Delta\chi$ for graphs with non-vanishing mean curvature vector $H = H(X,Z)$.

Let again $H_\sigma = H_\sigma(X,N_\sigma)$. Then it holds

$$\begin{aligned}\Delta\chi = &-2(2H^2-K)W\chi + 2\sum_{\sigma=1}^{n}(H_{\sigma,X}\cdot X_u + H_{\sigma,Z}\cdot N_{\sigma,u})(n_\sigma^1 x_v^2 - n_\sigma^2 x_v^1) \\ &-2\sum_{\sigma=1}^{n}(H_{\sigma,X}\cdot X_v + H_{\sigma,Z}\cdot N_{\sigma,v})(n_\sigma^1 x_u^2 - n_\sigma^2 x_u^1)\end{aligned}$$

if the normal bundle of the surface is flat. For an explicite calculation we refer to Fröhlich [64].

From this form it is finally possible to infer again a stability inequality of the form

$$\iint\limits_B |\nabla\varphi|\, dudv \ge \mu \iint\limits_B (2H^2 - K)W\varphi^2\, dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$ with a suitable stability factor $\mu > 0$.

13.5 Eigenvalue problems on manifolds

The number "2" in the stability inequality

$$\iint\limits_B |\nabla\varphi|^2\, dudv \ge 2 \iint\limits_B (-K)W\varphi^2\, dudv$$

for minimal surfaces $X: B \to \mathbb{R}^3$ actually represents the first eigenvalue of the spherical Laplace operator on the half sphere. In fact, a minimal graph is stable and its spherical image is contained in a half sphere. Thus the first eigenvalue of this Laplacian on the spherical image of the graph is greater than 2.

In the following we want to develop a basic theory of such eigenvalue problems, and we present some applications of this theory to problems concerning stability and μ-stability.

The Rayleigh quotient

Let a smooth and regular mapping $X\colon B \to \mathbb{R}^{n+2}$ with a line element

$$ds^2 = g_{11}\,du^2 + 2g_{12}\,du\,dv + g_{22}\,dv^2$$

be given. Its trace in space is a twodimensional set $M^2 \subset \mathbb{R}^{n+2}$ with smooth boundary $\partial M^2 \subset \mathbb{R}^{n+2}$ which we identify with the mapping $X = X(u,v)$.

Definition 13.4. The *Laplace operator Δ_{M^2} on the parametrically given surface $M^2 \subset \mathbb{R}^{n+2}$* is defined by

$$\Delta_{ds^2}\phi := \frac{1}{\sqrt{g_{11}g_{22}-g_{12}^2}}\left(\frac{\partial}{\partial u}\,\frac{g_{22}\phi_u - g_{12}\phi_v}{\sqrt{g_{11}g_{22}-g_{12}^2}} + \frac{\partial}{\partial v}\,\frac{g_{11}\phi_v - g_{12}\phi_v}{\sqrt{g_{11}g_{22}-g_{12}^2}}\right) \qquad (13.5)$$

for smooth functions $\phi \in C^2(B,\mathbb{R})$.

For real λ we now consider the eigenvalue problem

$$\Delta_{ds^2}\varphi + \lambda\varphi = 0 \quad \text{in } B, \qquad \varphi = 0 \quad \text{in } \partial B. \qquad (13.6)$$

The function $\varphi \in C_0^2(B,\mathbb{R})$ is called an eigenfunction to Δ_{M^2} if there is a number $\lambda \in \mathbb{R}$ such that (13.6) is satisfied. In this case the number λ is called an eigenvalue.

Multiplication of the eigenvalue equation (13.6) with φ followed by an integration by parts gives us

$$\lambda \iint\limits_B \varphi^2 W\,du\,dv = -\iint\limits_B \varphi\Delta_{ds^2}\varphi W\,du\,dv = \iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,du\,dv$$

with the area element $W = \sqrt{g_{11}g_{22}-g_{12}^2}$ and the Beltrami operator of first kind

$$\nabla_{ds^2}(\phi,\psi) = \sum_{i,j=1}^{2} g^{ij}\phi_{u^i}\psi_{u^j}$$

w.r.t. the line element ds^2 which we already met before. For such calculus rules regarding non-Euclidean differential operators we refer the reader to Blaschke and Leichtweiss [15].

Definition 13.5. The *Rayleigh quotient* is defined as

$$\mathscr{R}[\varphi] := \frac{\iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,du\,dv}{\iint\limits_B \varphi^2 W\,du\,dv}\,.$$

The infimum of the Rayleigh quotient corresponds exactly to the first eigenvalue of the Laplacian on $M^2 \subset \mathbb{R}^{n+2}$, i.e.

$$\lambda_1 := \inf_{\varphi \in V(B,\mathbb{R})} \mathscr{R}[\varphi]$$

on the function space

$$V(B,\mathbb{R}) := \left\{ \phi \in H^{1,2}(B,\mathbb{R}) \cap C^0(B,\mathbb{R}) \,:\, \phi \not\equiv 0,\ \phi = 0 \text{ on } \partial B \right\}.$$

Existence and regularity of the first eigenfunction

First of all we recall that there exist an eigenfunction $\varphi \in V(B,\mathbb{R})$ with eigenvalue $\lambda_1 > 0$ at all, see e.g. Bandle [5] or Sakai [138] for the following result.

Proposition 13.2. *The first eigenvalue λ_1 of (13.6) is positive, and it has multiplicity* 1, *i.e. up to sign there is exactly one eigenfunction φ_1 for λ_1. It particularly holds*

$$\varphi_1 > 0 \quad \textit{or} \quad \varphi_1 < 0 \quad \textit{in } B.$$

The uniqueness follows essentially from the usual maximum principle. Higher regularity $\varphi \in C_0^k(B,\mathbb{R})$ can be inferred from Weyl's lemma, see e.g. Hellwig [83] or Jost [103].

The first eigenvalue on spherical domains

As we will see immediately the first eigenvalue and the associated eigenfunction of the problem (13.6) on *spherical domains* can be employed successfully.

As usual we denote by S^2 the unit sphere. From Barbosa and do Carmo [8] we cite the following monotonicity results.

Proposition 13.3. *Let $\Omega \subset S^2$ and $\Theta \subset S^2$ be two simply connected, spherical compact domains with C^1-regular boundaries.*

1. *If $\Omega \subset \Theta$ then it holds*
$$\lambda_1(\Omega) \ge \lambda_1(\Theta).$$
2. *Let*
$$S^2_\omega := \left\{ Z \in S^2 \,:\, Z \cdot (0,0,1) \ge \cos\omega \right\}$$
be a spherical cap on S^2, and suppose that $\Omega \subset S^2$ satisfies
$$\text{Area}\,(\Omega) = \text{Area}\,(S^2_\omega).$$
Then it holds
$$\lambda_1(S^2_\omega) \le \lambda_1(\Omega)$$
with equality if and only if $\Omega = S^2_\omega$.

In Sato [139] we find some numerical values of the first eigenvalue $\lambda_1(S^2_\omega)$ for different angles ω which illustrate some of our later results. We present them here in rounded form.

ω	0.03	0.16	0.31	0.63	0.79	0.94
$\lambda_1(S^2_\omega)$	6422.2	225.39	59.73	14.24	8.92	6.20

ω	1.10	1.26	1.41	1.57	1.61	1.64
$\lambda_1(S^2_\omega)$	4.45	3.31	2.57	2.0	1.90	1.80

ω	1.68	1.72	1.77	1.82	1.87	1.93
$\lambda_1(S^2_\omega)$	1.70	1.60	1.50	1.40	1.30	1.20

ω	1.99	2.07	2.15	2.23	2.33	2.55
$\lambda_1(S^2_\omega)$	1.10	1.00	0.90	0.80	0.70	0.60

ω	2.57	2.71	2.87	3.03	3.13
$\lambda_1(S^2_\omega)$	0.50	0.40	0.30	0.20	0.10

The value $\omega = 1.57$ represents the half sphere $S^2_{\frac{\pi}{2}} \subset S^2$ on which λ_1 is exactly 2 (this completes our remarks on stable minimal graphs at the beginning of this section). Moreover there hold

$$\lim_{\omega \to 0_+} \lambda_1(S^2_\omega) = +\infty, \quad \lim_{\omega \to \pi_-} \lambda_1(S^2_\omega) = 0.$$

From the above tables it seems evident that λ_1 decreases monotonically as ω increases. In fact, for $\Omega \subset \Theta \subset S^2$ it holds the following *monotonicity property*

$$\lambda_1(\Omega) \geq \lambda_1(\Theta).$$

Furthermore, if $\text{Area}(\Omega) = \text{Area}(S^2_\omega)$ then

$$\lambda_1(\Omega) \geq \lambda_1(S^2_\omega).$$

In other words: *Symmetrical domains minimize the first eigenvalue.*

All these results can be found in more detail in the huge literature, e.g. Bandle [5], Barbosa and do Carmo [8] or Sakai [138]. We also want to refer to the classical work Courant and Hilbert [39].

The first eigenvalue and the Gauss curvature

We need some further comparison results for the first eigenvalue λ_1 on curved manifolds which we also take from Barbosa and do Carmo [8].

Let $M^2(K_0) \subset \mathbb{R}^{n+2}$ denote a smooth and regular surface with constant Gauss curvature $K_0 > 0$, and let a further surface $M^2 \subset \mathbb{R}^{n+2}$ with variable Gauss curvature be given.

Proposition 13.4. *Let $D \subset M^2$ be compact and simply connected, and let $K \leq K_0$ be the Gaussian curvature of M^2 with a real constant $K_0 > 0$. Furthermore, let $D^* \subset M^2(K_0)$ be a geodesic disc such that*

$$\text{Area}\,(D^*) = \text{Area}\,(D).$$

Then it holds

$$\lambda_1(D) \geq \lambda_1(D^*).$$

In other words, the first eigenvalue is minimized on constantly curved manifolds.

Proposition 13.5. *Let $D \subset (M^2, ds^2)$ be compact and simply connected with $M^2 \subset \mathbb{R}^{n+2}$ being a smooth, simply connected and regular surface with metric ds^2, and let ψ be a non-negative C^2-function on M^2 which vanishes in at most isolated points. Denote by $\lambda_1 > 0$ the first eigenvalue to the problem*

$$\triangle_{ds^2}\varphi + \lambda\varphi = 0 \quad \text{in } D, \quad \varphi = 0 \quad \text{on } \partial D.$$

Now consider on M^2 the new metric $d\hat{s}^2 := \psi\, ds^2$. Suppose that the Gauss curvature $\widehat{K}$ of $(M, d\hat{s}^2)$ satisfies

$$\widehat{K} \leq K_0$$

with a real constant $K_0 > 0$. Then it holds

$$\lambda_1 \geq \lambda_1(D^*)$$

with $D^* \subset M^2(K_0)$ being a geodesic disc with the same area as D in $(M, d\hat{s}^2)$.

13.6 μ-stability due to Ruchert, Barbosa and do Carmo

Proving the existence of a lower bound $\chi_0 > 0$ for the Jacobian of the plane mapping $(x^1(u,v), x^2(u,v))$ in the form

$$0 < \chi_0 \leq \chi = \frac{x_u^1 x_v^2 - x_v^1 x_u^2}{W}$$

as supposed in Theorem 13.4 is a rather difficult problem. In case $n = 1$ of one codimension this assumption demands that the third component N^3 of the unit normal vector of the minimal graph is greater than $\chi_0 > 0$.

In this section we want to prove a further stability result which comprises spectral methods from Barbosa and do Carmo [8] und Ruchert [137].

Theorem 13.5. *Let the conformally parametrized minimal surface X with a torsion-free ONF N be given. For a real number* $\kappa_0 > 0$ *assume that*

$$Q := \iint_B (\kappa_0 - K)W\,dudv < \omega_0$$

with a real constant $\omega_0 \in (0, 4\pi)$*. Let* $S^2_\omega \subset S^2$ *denote the spherical cap of spherical angle* ω *such that*

$$\text{Area}\,(S^2_\omega) = \omega_0\,,$$

and let $\mu > 0$ *be the smallest eigenvalue of the spherical Laplacian* Δ_{S^2} *on* S^2_ω*. Then the surface X is* μ*-stable with this number* μ *in the following sense*

$$\iint_B |\nabla\varphi|^2\,dudv \ge \mu \iint_B (-K)W\varphi^2\,dudv$$

for all $\varphi \in C^\infty_0(B, \mathbb{R})$.

For the proof we recall some important facts from chapter 6: For the curvature

$$\chi\widehat{K} = K - \frac{1}{2W}\triangle\log\chi, \quad K = \frac{2}{W^2}\left\{\frac{W_w W_{\overline{w}}}{W} - W_{w\overline{w}}\right\},$$

of the Fubini-Study metric we already have proved $\widehat{K} \le 1$, see Theorem 6.3.

Proof of the theorem. Let $\widehat{\triangle}$ denote the Laplace-Beltrami operator w.r.t. the metric $\widehat{g}_{ij} := \chi g_{ij}$. Furthermore, $\widehat{\lambda}_1$ means the first eigenvalue to the problem

$$\widehat{\triangle}\varphi + \lambda\varphi = 0 \quad \text{in } B, \quad \varphi = 0 \quad \text{on } \partial B,$$

and let $\lambda^*_1 > 0$ be the first eigenvalue of

$$\triangle^*\varphi^* + \lambda^*\varphi^* = 0 \quad \text{in } S^2_\omega\,, \quad \varphi^* = 0 \quad \text{on } \partial S^2_\omega$$

with a spherical cap $S^2_\omega \subset S^2$ such that Area$\,(S^2_\omega) = Q$. Since $\widehat{K} \le 1$ we infer $\lambda^*_1 \le \widehat{\lambda}_1$, and by our assumption it holds

$$\mu < \lambda^*_1 \le \widehat{\lambda}_1 \le \frac{\displaystyle\iint_B |\nabla\varphi|^2\,dudv}{\displaystyle\iint_B (\kappa_0 - K)W\varphi^2\,dudv}$$

for any $\varphi \in C^\infty_0(B, \mathbb{R})$.

It follows

$$\iint\limits_B |\nabla\varphi|^2\,dudv > \mu\iint\limits_B (\kappa_0 - K)W\varphi^2\,dudv > \mu\iint\limits_B (-K)W\varphi^2\,dudv,$$

and the statement is proved. □

We want to clarify the connection of our theorem with a theorem from Barbosa and do Carmo [8]:

Theorem 13.6. *For a real $a \in (0,2]$ assume that $\widehat{K} \le a$ for the Gaussian curvature of the spherical metric from our theorem above. Then a minimal surface is stable with $\mu = 2$ (with no curvature restrictions on the normal bundle) if*

$$\iint\limits_B (-K)W\,dudv \le \frac{4\pi}{1+a}\,.$$

In fact, Barbosa and do Carmo proved

$$\widehat{K} \le 2$$

as we already mentioned in chapter 6. *Our result is thus a improvement under the assumption of flat normal bundle where we get a sharper bound on $\widehat{K}$.*

Further stability results for weighted minimal surfaces in $\mathbb{R}^3$ using these methods from the spectral theory of the Laplacian on manifolds are presented in chapter 15.

To conclude this section we want to refer to Spruck [150] who estabished a stability criteria for minimal surfaces with sufficiently small curvatura integra using a generalized Sobolev inequality from Michael and Simon [119].

13.7 Calibration forms

We briefly and incompletely discuss the method of calibration forms for minimal surfaces.

Let $X = (x,y,\zeta(x,y))$ be a minimal graph in $\mathbb{R}^3$, i.e. ζ satisfies the minimal surface equation (see chapter 9)

$$\operatorname{div}\left(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\right) = 0 \quad \text{in } \Omega \subset \mathbb{R}^2\,.$$

Now let $\widetilde{\zeta}$ be a comparison function such that $\widetilde{\zeta} = \zeta$ on the boundary $\partial\Omega$. It holds

$$\iint\limits_\Omega \frac{\nabla\zeta\cdot\nabla(\zeta-\widetilde{\zeta})}{\sqrt{1+|\nabla\zeta|^2}} = 0.$$

This follows after partial integration taking $(\widetilde{\zeta}-\zeta)|_{\partial\Omega}$ into account. Thus we can estimate

$$\begin{aligned}\iint\limits_{\Omega}\sqrt{1+|\nabla\zeta|^2}\,dxdy &= \iint\limits_{\Omega}\frac{1+|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}}\,dxdy\\ &= \iint\limits_{\Omega}\frac{1+\nabla\zeta\cdot\nabla\widetilde{\zeta}}{\sqrt{1+|\nabla\zeta|^2}}\\ &\le \iint\limits_{\Omega}\sqrt{1+|\nabla\widetilde{\zeta}|^2}\,dxdy\end{aligned}$$

because

$$1+\nabla\zeta\cdot\nabla\widetilde{\zeta}\le\sqrt{1+|\nabla\zeta|^2}\sqrt{1+|\nabla\widetilde{\zeta}|^2}\,,$$

see Giusti [72], section 13.7.

But what happens in case of higher codimension?

Let us consider exemplarily the special case $n=2$ of minimal surfaces in $\mathbb{R}^4$. On the manifold $M^2\subset\mathbb{R}^4$, regarded as the image under the mapping $X\colon B\longrightarrow\mathbb{R}^4$, we consider a 2-form $\omega=\omega(\mathscr{X},\mathscr{Y})$ for tangent vector fields $\mathscr{X}$ and $\mathscr{Y}$. Such a form is defined as an alternating and differentiable mapping on the manifold such that

1. $\omega(\mathscr{X}+\mathscr{Y},\mathscr{Z})=\omega(\mathscr{X},\mathscr{Z})+\omega(\mathscr{Y},\mathscr{Z})$;
2. $\omega(f\cdot\mathscr{X},\mathscr{Y})=f\cdot\omega(\mathscr{X},\mathscr{Y})$ for differentiable f;
3. $\omega(\mathscr{X},\mathscr{Y})=-\omega(\mathscr{Y},\mathscr{X})$.

Please consult the wide literature for detailed discussions on differential forms, e.g. Blaschke and Leichtweiss [15], Heil [78] or Cartans original work [23].

Next, for complex valued vectors $x,y\in\mathbb{C}^2$ we define the *Kähler form*

$$\omega(x,y)=\operatorname{Im}\sum_{j=1}^{2}\overline{x^j}y^j\,.$$

The point is that this form satisfies

$$\omega(X_u/\sqrt{W},X_v/\sqrt{W})=1$$

if and only if $X=(\Phi(w),\Psi(w))$ is conformally parametrized, i.e. $X_u/\sqrt{W}$ and $X_v/\sqrt{W}$ is an orthonormal basis of the tangent space, and Φ and Ψ are holomorphic functions. Otherwise it holds $\omega\le 1$; remember particularly our example $X(w)=(w,w^2)$!

A differential form with these properties is called a calibration form.

Now denoting by $X^*\omega$ the *pull back* of ω (defined on the surface) onto the domain B by the mapping X, we compute using Stokes' theorem and Poincare's lemma (identify $X : B \to \mathbb{R}^4$ with the imbedding of the image $X(B)$ into $\mathbb{R}^4$)

$$\mathscr{A}[X] = \int_B X^* 1 = \int_B X^* \omega = \int_B d(X^* \tau) = \int_{\partial B} X^* \tau = \int_{\partial B} \widetilde{X}^* \tau = \int_B \widetilde{X}^* \omega \le \mathscr{A}[\widetilde{X}]$$

under the assumption that $\widetilde{X} = X$ on ∂B.

Theorem 13.7. *Let $X = (\Phi, \Psi)$ with Φ and Ψ holomorphic functions in B. Then X minimizes the area among all mappings $\widetilde{X} : B \to \mathbb{R}^4$ with the property $X(w) = \widetilde{X}(w)$ for all $w \in \partial B$.*

These calculations are taken from Eschenburg and Jost [52]. See also the mathematics resource website Wolfram MathWorld for a good introduction to Kähler forms and calibration forms.

Chapter 14
Energy estimates

In this chapter we establish various estimates for the area of surfaces with prescribed mean curvature and for their spherical energies. We mainly consider μ-stable immersions and surface graphs.

14.1 Geodesic discs

We have frequently used *geodesic discs:* Assume the immersion $X: B \to \mathbb{R}^{n+2}$ (or a part of it) represents a geodesic disc $\mathfrak{B}_r(X_0)$ of geodesic radius $r > 0$ about the center $X_0 \in \mathbb{R}^{n+2}$. Using geodesic polar coordinates ρ and φ we get a reparametrization of the form

$$X = X(\rho, \varphi): [0, r] \times [0, 2\pi] \longrightarrow \mathbb{R}^{n+2}.$$

For the corresponding line element ds_p^2 we have

$$\begin{aligned} ds_p^2 &= X_\rho(\rho,\varphi)\,d\rho^2 + 2X_\rho(\rho,\varphi)\cdot X_\varphi(\rho,\varphi)\,d\rho\,d\varphi + X_\varphi(\rho,\varphi)^2\,d\varphi^2 \\ &= d\rho^2 + P(\rho,\varphi)\,d\varphi^2 \end{aligned} \tag{14.1}$$

with a continuously differentiable function P satisfying the properties

$$P(\rho,\varphi) > 0 \quad \text{for all } (\rho,\varphi) \in (0,r] \times [0,2\pi) \tag{14.2}$$

as well as

$$\lim_{\rho \to 0_+} P(\rho,\varphi) = 0, \quad \lim_{\rho \to 0_+} \frac{\partial}{\partial \rho} \sqrt{P(\rho,\varphi)} = 1 \quad \text{for all } \varphi \in [0,2\pi). \tag{14.3}$$

For a comprehensive introduction of geodesic discs we want to refer the reader to Klingenberg [106]. For our purposes the following remarks are already sufficient:

1. For immersions with non-positive Gaussian curvature

$$K \le 0,$$

the so-called *exponential map,* which sends tangential vectors $V \in \mathfrak{T}_X(w)$ injectively onto geodesic curves on the surface, *is globally injective.* This particularly means, given a simply-connected and complete surface without boundary, we can introduce geodesic discs for *arbitrary* $r > 0$.

2. I some geodesic curve on a surface with positive Gaussian curvature

$$K \geq K_0 > 0$$

has length greater than $\frac{\pi}{\sqrt{K_0}}$ *then the exponential map is not injectiv.* Thus

$$\frac{\pi}{\sqrt{K_0}}$$

is the supremum of our geodesic radius $r > 0$.

We particularly refer Klingenberg [106], Theorems 3.4.13 and 3.4.16.

14.2 The area of μ-stable geodesic discs

In the following we want to establish estimates for the area of geodesic discs which are additionally μ-stable in the following sense

$$\begin{aligned} &\iint_B \nabla_{ds^2}(\psi,\psi)W\,dudv \\ &\quad \geq \mu \iint_B (q-K)W\psi^2\,dudv - \alpha(n)\sum_{i,j=1}^{2}\sum_{\sigma,\vartheta=1}^{n}\iint_B g^{ij}T_{\sigma,i}^{\vartheta}T_{\sigma,j}^{\vartheta}W\psi^2\,dudv \end{aligned} \tag{14.4}$$

for all $\psi \in C_0^\infty(B,\mathbb{R})$ and with a real number $\alpha(n) \geq 0$, see formula (13.3).

Using geodesic polar coordinates ρ and φ it takes the form

$$\begin{aligned} &\int_0^r\int_0^{2\pi} \nabla_{ds_P^2}(\psi,\psi)\sqrt{P(\rho,\varphi)}\,d\rho d\varphi \\ &\quad = \int_0^r\int_0^{2\pi} \frac{P(\rho,\varphi)\psi_\rho(\rho,\varphi)^2+\psi_\varphi(\rho,\varphi)^2}{P(\rho,\varphi)}\sqrt{P(\rho,\varphi)}\,d\rho d\varphi \\ &\quad \geq \mu\int_0^r\int_0^{2\pi}(q-K)\psi(\rho,\varphi)^2\sqrt{P(\rho,\varphi)}\,d\rho d\varphi \\ &\qquad -\alpha(n)\sum_{\sigma,\vartheta=1}^{n}\int_0^r\int_0^{2\pi}\Big\{P(\rho,\varphi)(T_{\sigma,1}^{\vartheta})^2+(T_{\sigma,2}^{\vartheta})^2\Big\}\psi(\rho,\varphi)^2\,d\rho d\varphi. \end{aligned}$$

The method of the proof of our first result follows essentially Gulliver [76] and Sauvigny [141].

Theorem 14.1. *Let the immersion X represent a geodesic disc $\mathfrak{B}_r(X_0) \subset \mathbb{R}^{n+2}$ of geodesic radius $r > 0$. Assume that X is μ-stable in the sense of (14.4) with real*

$$\mu > \frac{1}{2}\,.$$

Then it holds the area estimate

$$\mathscr{A}[X] \le \frac{1}{2\mu - 1}\left\{2\pi\mu + 2\alpha(n)\mathscr{T}_X[N]\right\}r^2 \tag{14.5}$$

with the total torsion $\mathscr{T}_X[N]$ of some chosen ONF N, i.e. the area of the geodesic disc growth quadratically in r.

Proof. 1. First we introduce geodesic polar coordinates (ρ,φ) as above. Then along curves $\rho = \text{const}$ the integral formula of Bonnet and Gauss can be written as

$$\int\limits_0^{2\pi} \kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\varphi + \int\limits_0^{\rho}\int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi = 2\pi$$

with the geodesic curvature κ_g along curves with $\rho = \text{const}$, see e.g. Blaschke and Leichtweiss [15]. Furthermore, from the same source [15], §81 we take

$$\kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)} = \frac{\partial}{\partial\rho}\sqrt{P(\rho,\varphi)}\,, \quad (\rho,\varphi) \in (0,r] \times [0,2\pi),$$

along such a curve, and we arrive at

$$\begin{aligned}\frac{\partial}{\partial\rho}\int\limits_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi &= \int\limits_0^{2\pi}\kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\varphi \\ &= 2\pi - \int\limits_0^{\rho}\int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi\end{aligned}$$

for all $\varphi \in (0,r]$.

2. We abbreviate the left hand side of the latter identity by

$$L(\rho) := \int\limits_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi, \quad \varphi \in (0,r].$$

For its first and second derivatives we compute

$$L'(\rho) = 2\pi - \int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi$$

as well as

$$L''(\varphi) = -\int_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\varphi.$$

3. Now consider the special test function

$$\Phi(\rho) := 1 - \frac{\rho}{r}, \quad \varphi \in (0,r],$$

satisfying

$$\nabla_{ds_P^2}(\Phi,\Phi) = \frac{P(\rho,\varphi)\Phi_\rho(\varphi)^2 + \Phi_\varphi(\rho)}{P(\rho,\varphi)} = \Phi'(\rho)^2$$

w.r.t. the line element ds_P^2 from (14.1). Here $\nabla_{ds_P^2}(\Phi,\Phi)$ denotes the Beltrami operator of first kind w.r.t. ds_P^2. We estimate as follows

$$\begin{aligned}
\int_0^r \Phi'(\rho)^2 L(\rho)\,d\rho &= \int_0^r\int_0^{2\pi} \Phi'(\rho)^2\sqrt{P(\rho,\varphi)}\,d\rho d\varphi \\
&\geq \mu \int_0^r\int_0^{2\pi} (q-K)\Phi(\rho)^2\sqrt{P(\rho,\varphi)}\,d\rho d\varphi \\
&\quad - \alpha(n)\sum_{\sigma,\vartheta=1}^n \int_0^r\int_0^{2\pi}\Big\{P(\rho,\varphi)(T_{\sigma,1}^\vartheta)^2 + (T_{\sigma,2}^\vartheta)^2\Big\}\,d\rho d\varphi \\
&\geq \mu \int_0^r L''(\rho)\Phi(\rho)^2\,d\rho \\
&\quad - \alpha(n)\sum_{\sigma,\vartheta=1}^n \int_0^r\int_0^{2\pi}\Big\{P(\rho,\varphi)(T_{\sigma,1}^\vartheta)^2 + (T_{\sigma,2}^\vartheta)^2\Big\}\,d\rho d\varphi
\end{aligned}$$

due the μ-stability condition. Partial integration together with (14.3) gives

$$\begin{aligned}
\int_0^r L''(\rho)\Phi(\rho)^2\,d\rho &= L'(\rho)\Phi(\rho)\Big|_{\rho=0_+}^{\rho=r} - 2\int_0^r L'(\rho)\Phi(\rho)\Phi'(\rho)\,d\rho \\
&= -2\pi - 2L(\rho)\Phi(\rho)\Phi'(\rho)\Big|_{\rho=0_+}^{\rho=r} + 2\int_0^r L(\rho)\Phi'(\rho)^2\,d\rho \\
&= -2\pi + 2\int_0^r L(\rho)\Phi'(\rho)^2\,d\rho.
\end{aligned}$$

Since $\Phi \in [0,1]$ we get

$$\begin{aligned}\int_0^r \Phi'(\rho)^2 L(\rho)\,d\rho \,\geq\, &-2\pi\mu + 2\mu \int_0^r \Phi'(\rho)^2 L(\rho)\,d\rho \\ &- \alpha(n) \sum_{\sigma,\vartheta=1}^{n} \int_0^r \int_0^{2\pi} \Big\{ P(\rho,\varphi)(T_{\sigma,1}^{\vartheta})^2 + (T_{\sigma,2}^{\vartheta})^2 \Big\}\, \Phi(\rho)^2\, d\rho d\varphi \\ \geq\, &-2\pi\mu + 2\mu \int_0^r \Phi'(\rho)^2 L(\rho)\,d\rho \\ &- \alpha(n) \sum_{\sigma,\vartheta=1}^{n} \int_0^r \int_0^{2\pi} \Big\{ P(\rho,\varphi)(T_{\sigma,1}^{\vartheta})^2 + (T_{\sigma,2}^{\vartheta})^2 \Big\}\, d\rho d\varphi.\end{aligned}$$

Rearranging thus yields

$$\begin{aligned}(2\mu - 1) &\int_0^r \Phi(\rho)^2 L(\rho)\,d\rho \\ &\leq 2\pi\mu + \alpha(n) \sum_{\sigma,\vartheta=1}^{n} \int_0^r \int_0^{2\pi} \Big\{ P(\rho,\varphi)(T_{\sigma,1}^{\vartheta})^2 + (T_{\sigma,2}^{\vartheta})^2 \Big\}\, d\rho d\varphi,\end{aligned}$$

and the statement follows taking

$$\frac{1}{r^2}\mathscr{A}[X] = \int_0^r \Phi'(\rho)^2 L(\rho)\,d\rho$$

into account. □

The question whether a similar estimate is true for values $\mu < \frac{1}{2}$ remains open. We refer the reader to a discussion in Fischer-Colbrie and Schoen [60].

14.3 An area estimate for minimal graphs

Following Bergner and Fröhlich [13] want to show how one could establish area bounds for minimal graphs independently of stability, total torsion, or other things.

Theorem 14.2. *Let $\mathfrak{B}_r(X_0)$ denote a geodesic disc of the minimal graph*

$$X(w) = (w, w^2), \quad w \in B_R := \big\{ w \in \mathbb{C}^2 : |w| \leq R \big\},$$

with radius $r > 0$ and centered at $X_0 := X(0,0)$.

Then it holds

$$\mathscr{A}[\mathfrak{B}_r(X_0)] \le 192\pi r^2 \quad \textit{for sufficiently large } r > 0. \tag{14.6}$$

Proof. Denote by

$$c(t) = \big(u(t), v(t)\big) \subset B_R\,, \quad t \in [0,T],$$

an arc-length parametrized and smooth curve with the properties

(i) $c(0) = (0,0)$ and $c(T) \in \partial B_R$;

(ii) $|\dot{c}(t)|^2 = 1$ for all $t \in [0,T]$.

The set of all such curves is abbreviated with $\mathfrak{C}$. Now let $c(t)$ be chosen from $\mathfrak{C}$. We will establish a lower bound for the length $\mathscr{L}[X \circ c]$ of its image on the minimal graph. W.l.o.g. we assume $|u(T)| \ge |v(T)|$. It particularly holds

$$|u(T)| \ge \frac{R}{2}\,.$$

We define

$$t^* := \sup\left\{ t \in (0,T) \,:\, |u(t)| \le \frac{R}{4} \right\}. \tag{14.7}$$

Note that $T - t^* \ge \frac{R}{4}$ for the special arc-length parametrization. We estimate now as follows:

$$\begin{aligned}
\mathscr{L}[X \circ c] &= \int_0^T \sqrt{1+4u^2+4v^2}\,dt \ge \int_0^T \sqrt{1+4u^2}\,dt \ge \int_{t^*}^T \sqrt{1+4u^2}\,dt \\
&\ge 2\int_{t^*}^T |u(t)|\,dt \ge \frac{R}{2}(T-t^*) \ge \frac{R^2}{8}\,.
\end{aligned}$$

The case $|v(T)| \ge |u(T)|$ can be treated analogously. Now let

$$r := \min_{c \in \mathfrak{C}} \mathscr{L}[X \circ c].$$

Due to the graphical property of the minimal surface, the geodesic disc $\mathfrak{B}_r(X_0)$ projects one-to-one into the disc B_R. Together with (14.7) we get

$$\begin{aligned}
\mathscr{A}[\mathfrak{B}_r(X_0)] &\le \iint_{B_R} (1+4u^2+4v^2)\,du\,dv = \int_0^{2\pi}\int_0^R (1+4\rho^2)\rho\,d\rho\,d\varphi \\
&= 2\pi\left(\frac{R^2}{2} + R^4\right) \le 3\pi R^4 \le 3\cdot 8^2\cdot \pi r^2 = 192\pi r^2
\end{aligned}$$

for all $R \ge 1$ proving the statement. $\square$

14.4 Area estimates via the curvatura integra

With the next result and its proof we follow again Sauvigny [141] and use the integral formula of Bonnet and Gauss to infer an area estimate in terms of the total curvature of the immersion.

Theorem 14.3. *Let the immersion X represent a geodesic disc $\mathfrak{B}_r(X_0)$ of radius $r>0$. Suppose that its Gaussian curvature K satisfies*

$$K(\rho,\varphi)\le K_0 \quad \textit{for all } (\rho,\varphi)\in(0,r]\times[0,2\pi)$$

with a real constant $K_0\in[0,+\infty)$. Then it holds

$$\mathscr{A}[X]\le r^2\left\{\pi+\frac{1}{2}\int_0^r\int_0^{2\pi}\Big\{K_0-K(\rho,\varphi)\Big\}\sqrt{P(\rho,\varphi)}\,d\rho\, d\varphi\right\} \tag{14.8}$$

for its area.

Proof. As in section 14.2 we consider curves $\rho=\text{const}$

$$\begin{aligned}\frac{\partial}{\partial\rho}\int_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi &= 2\pi-\int_0^\rho\int_0^{2\pi}K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau\, d\varphi\\ &\le 2\pi+\int_0^\rho\int_0^{2\pi}\Big\{K_0-K(\tau,\varphi)\Big\}\sqrt{P(\rho,\varphi)}\,d\tau\, d\varphi\\ &\le 2\pi+\int_0^r\int_0^{2\pi}\Big\{K_0-K(\tau,\varphi)\Big\}\sqrt{P(\tau,\varphi)}\,d\tau\, d\varphi.\end{aligned}$$

Integration over the radius ρ yields

$$\int_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi\le\rho\left\{2\pi+\int_0^r\int_0^{2\pi}\Big\{K_0-K(\tau,\varphi)\Big\}\sqrt{P(\tau,\varphi)}\,d\tau\, d\varphi\right\}.$$

A final integration over $\rho=0\dots r$ proves the statement. □

In particular, for minimal surfaces with $K_0=0$ we infer the estimate

$$\mathscr{A}[X]\le r^2\left\{\pi+\frac{1}{2}\int_0^r\int_0^{2\pi}\big\{-K(\rho,\varphi)\big\}\sqrt{P(\rho,\varphi)}\,d\rho\, d\varphi\right\}.$$

We immediately obtain an area estimate in terms of the total geodesic curvature κ_g of the boundary curve of the immersion X.

First it holds $|\kappa_g| \leq \kappa$ for the non-negative spatial curvature of the boundary. From the integral formula of Bonnet and Gauss we then obtain

$$\iint_B (-K)W\,dudv = \int_{\partial B} \kappa_g(s)\,ds - 2\pi \leq \int_{\partial B} \kappa(s)\,ds - 2\pi.$$

Inserting this inequality into the estimate of the previous theorem shows

$$\mathscr{A}[X] \leq \pi r^2 + \frac{K_0}{2}\mathscr{A}[X]\,r^2 - \pi r^2 + \frac{r^2}{2}\int_{\partial B} \kappa(s)\,ds.$$

Corollary 14.1. *It holds*

$$\mathscr{A}[X] \leq \frac{K_0}{2}\mathscr{A}[X]r^2 + \frac{r^2}{2}\int_{\partial B} \kappa(s)\,ds$$

with $K_0 \in [0,+\infty)$ *from Theorem 14.3.*

Consider again the case $K_0 = 0$ for a minimal surface. We infer

$$\mathscr{A}[X] \leq \frac{r^2}{2}\int_{\partial B} \kappa(s)\,ds.$$

In this special context we would like to mention Fenchel's theorem (see Fenchel [56]) which states

$$\int_{\partial B} \kappa(s)\,ds \geq 2\pi$$

with equality if and only if the boundary curve is a plane and convex curve.

14.5 The area of graphs with prescribed mean curvature

We continue our investigations on the Gulliver-type functional

$$\iint_\Omega \Gamma(x,y,\zeta)\sqrt{1+p^2+q^2+p^2q^2-(p\cdot q)^2}\,dxdy$$

for graphs $\zeta = \zeta(x,y)$ with the settings

$$\zeta = (\zeta_1,\ldots,\zeta_n), \quad p = \zeta_x \in \mathbb{R}^n\,,\ q = \zeta_y \in \mathbb{R}^n\,.$$

Our aim is to establish an upper bound for the area of critical points of the associated variational problem.

From Corollary 11.1 we knwo that critical points are solutions of

$$\begin{aligned}\operatorname{div}\frac{(p_\sigma,q_\sigma)}{W} &= 2H_\sigma(X,N_\sigma)\,\frac{\sqrt{1+p_\sigma^2+q_\sigma^2}}{W}\\ &+\frac{1}{\Gamma W}\left\{\left[p^2+q^2+p^2q^2-(p\cdot q)^2\right]\Gamma_{z_\sigma}-\sum_{\omega=1}^{n}(p_\sigma p_\omega+q_\sigma q_\omega)\Gamma_{z_\omega}\right\}\\ &-\frac{1}{\Gamma}\operatorname{div}\left(\frac{p_\sigma q^2-q_\sigma(p\cdot q)}{W}\Gamma,\frac{q_\sigma p^2-p_\sigma(p\cdot q)}{W}\Gamma\right)\end{aligned} \tag{14.9}$$

for all $\sigma=1,\ldots,n$ with the mean curvature field

$$H_\sigma(X,N_\sigma)=\frac{\Gamma_X(X)\cdot N_\sigma}{2\Gamma(X)}\,.$$

The case $\Gamma(X)\equiv 1$ represents the minimal surface case

$$\operatorname{div}\frac{(p_\sigma,q_\sigma)}{W}+\operatorname{div}\left(\frac{p_\sigma q^2-q_\sigma(p\cdot q)}{W},\frac{q_\sigma p^2-p_\sigma(p\cdot q)}{W}\right)=0\quad\text{in }\Omega \tag{14.10}$$

for all $\sigma=1,\ldots,n$. Moreover, the second divergence on the left hand side term vanishes identically if $n=1$.

In the following we suppose that Γ depends only on $(x,y)\in\Omega$, i.e.

$$\Gamma=\Gamma(x,y)\in C^1(\Omega,\mathbb{R}).$$

Theorem 14.4. *Let $\zeta=(\zeta_1,\ldots,\zeta_n)$ be a solution of (14.9) with a density function $\Gamma=\Gamma(x,y)\in C^1(\Omega,\mathbb{R})$. Let*

$$0<\Gamma_0\le\Gamma(x,y)\le\Gamma_1<+\infty\quad\text{in }\Omega,$$

$$\|\Gamma\|_{C^1(\Omega)}=:\Gamma_2<+\infty,$$

and assume furthermore

$$\Lambda:=1-\sqrt{2}\,n^2\,\frac{\Gamma_2}{\Gamma_0}\max_{\sigma=1,\ldots,n}\|\zeta_\sigma\|_{C^0(\Omega)}>0.$$

Then it holds

$$\begin{aligned}\Lambda\cdot\mathscr{A}[\zeta]\le|\Omega|+n|\partial\Omega|\max_{\sigma=1,\ldots,n}\|\zeta_\sigma\|_{C^0(\partial\Omega)}+2nh_0|\Omega|\max_{\sigma=1,\ldots,n}\|\zeta_\sigma\|_{C^0(\Omega)}\\ +\sqrt{2}\,n^2|\partial\Omega|\max_{\sigma=1,\ldots,n}\|\zeta_\sigma\|_{C^0(\partial\Omega)}\|D^\top\zeta_\sigma\|_{C^0(\partial\Omega)}\end{aligned} \tag{14.11}$$

with the circumference $|\partial\Omega|$ of the boundary $\partial\Omega\subset\mathbb{R}^2$, the area $|\Omega|$ of Ω, and the tangential derivative $D^\top\zeta_\sigma$ for $\sigma=1,\ldots,n$ along $\partial\Omega$.

Finally we set

$$h_0 := \max_{(x,y)\in\Omega} \max_{|N|=1} |H(x,y,N)|.$$

The second row in this area estimate does not appear in case of one codimension. We would also set $\Lambda := 1$ if $n = 1$.

The special situation for minimal graphs in the contents of the

Corollary 14.2. *Let ζ solve the minimal surface system (14.10). Then it holds*

$$\begin{aligned}\mathscr{A}[\zeta] \le |\Omega| + n|\partial\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\partial\Omega)} \\ + \sqrt{2}\,n^2|\partial\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\partial\Omega)} \|D^\top\zeta_\sigma\|_{C^0(\partial\Omega)}\,.\end{aligned} \tag{14.12}$$

We want to point out the dependence of the C^1-data of the solution in contrast to the case $n = 1$ where it holds

$$\mathscr{A}[\zeta] \le |\Omega| + c|\partial\Omega| \|\zeta\|_{C^0(\partial\Omega)}$$

with some suitable constant $c > 0$. For illustration consider the minimal graph $(w, w^n) \in \mathbb{R}^4$: For all $n \in \mathbb{N}$ it holds $\|\zeta_\sigma\|_{C^0(B)} = \sqrt{2}$ *independently of the area actually enclosed.*

Now we come to the proof of our theorem.

Proof. 1. First we sum up the n identities

$$\zeta_\sigma \operatorname{div} \frac{\nabla\zeta_\sigma}{W} = \operatorname{div} \frac{\zeta_\sigma \nabla\zeta_\sigma}{W} - \frac{|\nabla\zeta_\sigma|^2}{W}\,, \quad \sigma = 1,\ldots,n,$$

to get

$$\begin{aligned}\sum_{\sigma=1}^n \zeta_\sigma \operatorname{div} \frac{\nabla\zeta_\sigma}{W} &= \sum_{\sigma=1}^n \operatorname{div} \frac{\zeta_\sigma\nabla\zeta_\sigma}{W} - \frac{1}{W} \sum_{\sigma=1}^n (p_\sigma^2 + q_\sigma^2) \\ &= \sum_{\sigma=1}^n \operatorname{div} \frac{\zeta_\sigma\nabla\zeta_\sigma}{W} - \frac{1}{W}\,(p^2+q^2).\end{aligned}$$

For the area element $W = \sqrt{1+p^2+q^2+p^2q^2-(p\cdot q)^2}$ we compute

$$\begin{aligned}\frac{1}{W} - W &= \frac{1-\left[1+p^2+q^2+p^2q^2-(p\cdot q)^2\right]}{W} \\ &= -\frac{p^2+q^2}{W} - \frac{p^2q^2-(p\cdot q)^2}{W}\,.\end{aligned}$$

This yields

$$W = \frac{1}{W} + \sum_{\sigma=1}^n \operatorname{div} \frac{\zeta_\sigma\nabla\zeta_\sigma}{W} - \sum_{\sigma=1}^n \zeta_\sigma \operatorname{div} \frac{\nabla\zeta_\sigma}{W} + \frac{p^2q^2-(p\cdot q)^2}{W}\,. \tag{14.13}$$

Now we insert the Euler-Lagrange equations from (14.9). First multiply all these equations by ζ_σ and sum up to get (note $\Gamma_{z_\sigma} \equiv 0$)

$$\begin{aligned}
\sum_{\sigma=1}^{n} \zeta_\sigma \operatorname{div} \frac{\nabla \zeta_\sigma}{W} &= 2 \sum_{\sigma=1}^{n} H_\sigma(x,y,N_\sigma)\zeta_\sigma \frac{\sqrt{1+p_\sigma^2+q_\sigma^2}}{W} \\
&\quad - \sum_{\sigma=1}^{n} \zeta_\sigma \operatorname{div} \left(\frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{W}, \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{W} \right) \\
&\quad - \sum_{\sigma=1}^{n} \left\{ \frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{\Gamma(x,y)W} \zeta_\sigma \Gamma_x(x,y) + \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{\Gamma(x,y)W} \zeta_\sigma \Gamma_y(x,y) \right\}.
\end{aligned}$$

In the second row we take into account

$$\begin{aligned}
&\operatorname{div} \left(\frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{W} \zeta_\sigma, \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{W} \zeta_\sigma \right) \\
&\qquad = \zeta_\sigma \operatorname{div} \left(\frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{W}, \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{W} \right) \\
&\qquad\quad + \frac{p_\sigma^2 q^2 - p_\sigma q_\sigma (p\cdot q)}{W} + \frac{q_\sigma^2 p^2 - p_\sigma q_\sigma (p\cdot q)}{W},
\end{aligned}$$

and after summation

$$\begin{aligned}
&\sum_{\sigma=1}^{n} \zeta_\sigma \operatorname{div} \left(\frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{W}, \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{W} \right) \\
&\qquad = \sum_{\sigma=1}^{n} \operatorname{div} \left(\frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{W} \zeta_\sigma, \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{W} \zeta_\sigma \right) \\
&\qquad\quad - \frac{2}{W} \left\{ p^2 q^2 - (p\cdot q)^2 \right\}.
\end{aligned}$$

2. Now (14.13) can be written in the form

$$\begin{aligned}
W &= \frac{1}{W} + \sum_{\sigma=1}^{n} \operatorname{div} \frac{\zeta_\sigma \nabla \zeta_\sigma}{W} - \frac{p^2 q^2 - (p\cdot q)^2}{W} \\
&\quad - 2 \sum_{\sigma=1}^{n} H_\sigma(x,y,N_\sigma)\zeta_\sigma \frac{\sqrt{1+p_\sigma^2+q_\sigma^2}}{W} \\
&\quad + \sum_{\sigma=1}^{n} \operatorname{div} \left(\frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{W} \zeta_\sigma, \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{W} \zeta_\sigma \right) \\
&\quad + \sum_{\sigma=1}^{n} \left\{ \frac{p_\sigma q^2 - q_\sigma (p\cdot q)}{\Gamma(x,y)W} \zeta_\sigma \Gamma_x(x,y) + \frac{q_\sigma p^2 - p_\sigma (p\cdot q)}{\Gamma(x,y)W} \zeta_\sigma \Gamma_y(x,y) \right\}.
\end{aligned} \tag{14.14}$$

Then we can integrate as follows.

(i) We start with

$$\iint\limits_{\Omega} \frac{1}{W}\,dxdy \le |\Omega| \quad \text{because} \quad \frac{1}{W} \le 1. \tag{14.15}$$

(ii) Next for the second term we have

$$\begin{aligned}\sum_{\sigma=1}^{n} \iint\limits_{\Omega} \operatorname{div} \frac{\zeta_\sigma \nabla \zeta_\sigma}{W}\,dxdy &\le \sum_{\sigma=1}^{n} \int\limits_{\partial\Omega} \frac{|\nabla \zeta_\sigma \cdot \nu|}{W}\,|\zeta_\sigma|\,ds \\ &\le n|\partial\Omega| \max_{\sigma=1,\ldots,n-2} \|\zeta_\sigma\|_{C^0(\partial\Omega)}\end{aligned} \tag{14.16}$$

with the outer unit normal vector ν at the boundary $\partial\Omega$, taking

$$\frac{\sqrt{1+p_\sigma^2+q_\sigma^2}}{W} \le 1 \quad \text{for all} \sigma = 1,\ldots,n$$

into account.

(iii) The third term is non-positive and can be estimated by 0 :

$$-\frac{p^2q^2-(p\cdot q)^2}{W} \le 0.$$

(iv) Likewise it holds

$$\begin{aligned}&2\sum_{\sigma=1}^{n} \iint\limits_{\Omega} H_\sigma(x,y,N_\sigma)\zeta_\sigma \frac{\sqrt{1+p_\sigma^2+q_\sigma^2}}{W}\,dxdy \\ &\qquad \le 2nh_0|\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\Omega)}\,.\end{aligned} \tag{14.17}$$

(v) Now we come back to the second row in (14.14). Note that

$$\begin{aligned}p_\sigma q^2 - q_\sigma(p\cdot q) &= \sum_{\vartheta=1}^{n} (p_\sigma q_\vartheta - p_\vartheta q_\sigma)q_\vartheta\,, \\ q_\sigma p^2 - p_\sigma(p\cdot q) &= \sum_{\vartheta=1}^{n} (q_\sigma p_\vartheta - q_\vartheta p_\sigma)p_\vartheta\,.\end{aligned} \tag{14.18}$$

After multiplication by p_σ or q_σ and following summation we get

$$p^2q^2-(p\cdot q)^2 = \frac{1}{2}\sum_{\sigma,\vartheta=1}^{n} (p_\sigma q_\vartheta - p_\vartheta q_\sigma)^2\,. \tag{14.19}$$

Thus we estimate as follows:

$$
\begin{aligned}
&\sum_{\sigma=1}^{n} \iint\limits_{\Omega} \operatorname{div}\left(\frac{p_\sigma q^2 - q_\sigma(p\cdot q)}{W}\,\zeta_\sigma, \frac{q_\sigma p^2 - p_\sigma(p\cdot q)}{W}\,\zeta_\sigma\right) dxdy \\
&\quad = \sum_{\sigma,\omega=1}^{n} \iint\limits_{\Omega} \operatorname{div}\left(\frac{(p_\sigma q_\omega - p_\omega q_\sigma)q_\omega}{W}\,\zeta_\sigma, -\frac{(p_\sigma q_\omega - p_\omega q_\sigma)p_\omega}{W}\,\zeta_\sigma\right) dxdy \\
&\quad = \sum_{\sigma,\omega=1}^{n} \int\limits_{\partial\Omega} \frac{(p_\sigma q_\omega - p_\omega q_\sigma)\zeta_\sigma}{W}\,(q_\omega, -p_\omega)\cdot \nu\, ds \\
&\quad \le \sqrt{2} \sum_{\sigma,\omega=1}^{n} \int\limits_{\partial\Omega} |\zeta_\sigma||D^\top \zeta_\omega|\, ds \\
&\quad \le \sqrt{2}\,n^2 |\partial\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\partial\Omega)} \||D^\top\zeta_\sigma\|_{C^0(\partial\Omega)}\,.
\end{aligned}
\tag{14.20}
$$

(vi) To estimate the last terms we come back to (14.18), (14.19) and compute

$$
\begin{aligned}
\frac{|p_\sigma q^2 - q_\sigma(p\cdot q)|}{W} &\le \sum_{\vartheta=1}^{n} \frac{|p_\sigma q_\vartheta - q_\sigma p_\vartheta||q_\vartheta|}{W} \\
&\le \sqrt{2} \sum_{\vartheta=1}^{n} \frac{\sqrt{p^2q^2 - (p\cdot q)^2}}{W}\,|q_\vartheta|,
\end{aligned}
$$

and analogously

$$
\frac{|q_\sigma p^2 - p_\sigma(p\cdot q)|}{W} \le \sqrt{2} \sum_{\vartheta=1}^{n} \frac{\sqrt{p^2q^2 - (p\cdot q)^2}}{W}\,|p_\vartheta|.
$$

Thus we arrive at

$$
\begin{aligned}
&\sum_{\sigma=1}^{n} \iint\limits_{\Omega} \left\{\frac{p_\sigma q^2 - q_\sigma(p\cdot q)}{\Gamma(x,y)W}\,\zeta_\sigma \Gamma_x(x,y) + \frac{q_\sigma p^2 - p_\sigma(p\cdot q)}{\Gamma(x,y)W}\,\zeta_\sigma\Gamma_y(x,y)\right\} dxdy \\
&\quad \le \sum_{\sigma=1}^{n} \Gamma_2 \iint\limits_{\Omega} |\zeta_\sigma| \left\{\frac{|p_\sigma q^2 - q_\sigma(p\cdot q)|}{\Gamma(x,y)W} + \frac{|q_\sigma p^2 - p_\sigma(p\cdot q)|}{\Gamma(x,y)W}\right\} dxdy \\
&\quad \le \frac{\sqrt{2}\,n^2\Gamma_2}{\Gamma_0} \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(B)} \iint\limits_{\Omega} \sqrt{1 + p^2 + q^2 + p^2q^2 - (p\cdot q)^2}\, dxdy.
\end{aligned}
\tag{14.21}
$$

For this estimate we explicitely need the assumption $\Gamma = \Gamma(x,y)$. Otherwise there remain derivatives p_ϑ and q_ϑ quadratically and we are not able to estimates the area element.

3. Putting (14.15), (14.16), (14.17), (14.20) and (14.21) together we conclude

$$\begin{aligned}\mathscr{A}[\zeta] \le |\Omega| + n|\partial\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\partial\Omega)} + 2nh_0|\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\Omega)} \\ + \sqrt{2}\,(n-2)^2|\partial\Omega| \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\partial\Omega)} \|D^\top \zeta_\sigma\|_{C^0(\partial\Omega)} \\ + \frac{\sqrt{2}\,n^2\Gamma_2}{\Gamma_0} \max_{\sigma=1,\ldots,n} \|\zeta_\sigma\|_{C^0(\Omega)} \mathscr{A}[\zeta].\end{aligned}$$

Rearranging proves the statement. □

14.6 The isoperimetric inequality

Following ideas of Hurwitz [98] we want to give a proof of the isoperimetric inequality for immersions $X\colon B \to \mathbb{R}^{n+2}$ with prescribed mean curvature fields. We also want to refer the reader to Blaschke and Leichtweiss [15], Barbosa and do Carmo [7], or Osserman's review article [129].

Theorem 14.5. *Let the immersion $X\colon B \to \mathbb{R}^{n+2}$ with prescribed mean curvature field $H(X,N)$ and spanning a C^1-regular boundary curve $\Gamma \in \mathbb{R}^{n+2}$ be given. Assume that*

$$\sup_{X\in\mathbb{R}^{n+2}} \max_{|N|=1} H(X,N) \le h_0 \in [0,+\infty).$$

Suppose furthermore that the following smallness condition

$$1 - nh_0\|X - \overline{X(\partial B)}\|_{C^0(B)} > 0$$

is true with the mean value

$$\overline{X(\partial B)} := \int_0^{2\pi} Z(\varphi)\,d\varphi,$$

with a representation $Z = Z(\varphi)$ for the boundary curve $\Gamma \subset \mathbb{R}^{n+2}$ of the surface as described in the proof below. Then it holds the isoperimetric inequality

$$\mathscr{A}[X] \le \frac{1}{4\pi\big(1 - nh_0\|X - \overline{X(\partial B)}\|_{C^0(B)}\big)}\,\mathscr{L}[\Gamma]^2 \tag{14.22}$$

with the length $\mathscr{L}[\Gamma]$ of $\Gamma \subset \mathbb{R}^{n+2}$.

Proof. Let $(u,v) \in B$ be conformal parameters. We introduce geodesic polar coordinates to get the mapping

$$Y(r,\vartheta) = X(r\cos\vartheta, r\sin\vartheta).$$

Let A be a fixed vector. We compute

$$\begin{aligned}
\mathscr{A}[X] &= \frac{1}{2}\iint\limits_B \Big\{|X_u|^2+|X_v|^2\Big\}\,dudv \\
&= \frac{1}{2}\iint\limits_B \Big\{\big[(X-A)\cdot X_u\big]_u+\big[(X-A)\cdot X_v\big]_v\Big\}\,dudv-\frac{1}{2}\iint\limits_B (X-A)\cdot\Delta X\,dudv \\
&= \frac{1}{2}\int\limits_{\partial B}\big[(X-A)\cdot X_u,(X-A)\cdot X_v\big]\cdot\nu\,ds+\frac{1}{2}\iint\limits_B (X-A)\cdot\Delta X\,dudv \\
&\le \frac{1}{2}\int\limits_0^{2\pi}\big\{Y(1,\vartheta)-A\big\}\cdot Y_r(1,\vartheta)\,d\vartheta+\frac{1}{2}\iint\limits_B |X-A||\Delta X|\,dudv
\end{aligned}$$

with the outer unit normal vector $\nu=(\cos\vartheta,\sin\vartheta)$. Make additionally use of

$$X_u=\cos\vartheta\,Y_r-\frac{1}{r}\sin\vartheta\,Y_\vartheta\,,\quad X_v=\sin\vartheta\,Y_r+\frac{1}{r}\cos\vartheta\,Y_\vartheta\,.$$

From the estimate

$$|\Delta X|=\left|2\sum_{\sigma=1}^n H(X,N_\sigma)WN_\sigma\right|\le nh_0|\nabla X|^2$$

we infer

$$\mathscr{A}[X]\le\frac{1}{2}\int\limits_0^{2\pi}\big\{Y(1,\vartheta)-A\big\}\cdot Y_r(1,\vartheta)\,d\vartheta+\frac{nh_0}{2}\iint\limits_B |X-A||\nabla X|^2\,dudv.$$

Now let the curve $\Gamma\subset\mathbb{R}^{n+2}$ presented in arc-length via

$$Z(\varphi)+A\colon[0,2\pi]\longrightarrow\Gamma,\quad |Z'(\varphi)|=\frac{\mathscr{L}[\Gamma]}{2\pi}\quad\text{for all }\varphi\in[0,2\pi)$$

where we choose A such that for $Z=Z(\varphi)$ it holds a Fourier expansion in the following form

$$Z(\varphi)=\sum_{k=1}^\infty\big(A_k\cos k\varphi+B_k\sin k\varphi\big)$$

with vectors A_k and B_k, $k=1,2,\ldots$ We arrive at

$$\int\limits_0^{2\pi}\Big\{|Z'(\varphi)|^2-|Z(\varphi)|^2\Big\}\,d\varphi=\pi\sum_{k=1}^\infty(k^2-1)(A_k^2+B_k^2)\ge 0. \tag{14.23}$$

Then it follows that

$$\mathscr{A}[X] \leq \frac{1}{2}\int\limits_0^{2\pi} |Y(1,\vartheta)-A||Y_\vartheta(1,\vartheta)|\,d\vartheta + \frac{nh_0\|X-\overline{X(\partial B)}\|_{C^0(B)}}{2}\iint\limits_B |\nabla X|^2\,dudv$$
$$\leq \frac{1}{2}\int\limits_0^{2\pi} |Z(\varphi)||Z'(\varphi)|\,d\varphi + nh_0\|X-\overline{X(\partial B)}\|_{C^0(B)}\iint\limits_B W\,dudv$$

We continue as follows:

$$\begin{aligned}
&\mathscr{L}[\Gamma]^2 - 4\pi\mathscr{A}[X] \\
&= \mathscr{L}[\Gamma]^2 - 2\pi\int\limits_0^{2\pi} |Z(\varphi)||Z'(\varphi)|\,\varphi - 4\pi nh_0\|X-\overline{X(\partial B)}\|_{C^0(B)}\iint\limits_B W\,dudv \\
&= 2\pi\int\limits_0^{2\pi}\big(|Z'(\varphi)|^2 - |Z(\varphi)||Z'(\varphi)|\big)\,d\varphi - 4\pi nh_0\|X-\overline{X(\partial B)}\|_{C^0(B)}\iint\limits_B W\,dudv \\
&\geq \pi\int\limits_0^{2\pi}\big(|Z'(\varphi)|^2 - |Z(\varphi)|^2\big)\,d\varphi - 4\pi nh_0\|X-\overline{X(\partial B)}\|_{C^0(B)}\iint\limits_B W\,dudv \\
&\geq -4\pi nh_0\|X-\overline{X(\partial B)}\|_{C^0(B)}\iint\limits_B W\,dudv
\end{aligned}$$

using Cauchy's inequality and (14.23). After rearranging we have

$$\mathscr{L}[\Gamma]^2 \geq 4\pi(1-nh_0\|X-\overline{X(\partial B)}\|_{C^0(\partial B)})\mathscr{A}[X]$$

proving the statement. □

Corollary 14.3. *Let the minimal immersion $X\colon B\to\mathbb{R}^{n+2}$ be given, spanning the C^1-regular boundary curve $\Gamma\subset\mathbb{R}^3$. Then it holds the isoperimetric inequality*

$$\mathscr{A}[X] \leq \frac{1}{4}\mathscr{L}[\Gamma]^2 \tag{14.24}$$

with the length $\mathscr{L}[\Gamma]$ of $C\subset\mathbb{R}^3$.

Historical remarks on the isoperimetric inequality for minimal surfaces

The first successful approach in proving the isoperimetric inequality in the plane goes back to the Swiss mathematician Jacob Steiner (*1796 Utzendorf/Kt. Bern, †1863 Bern).

Though Steiner's proof is incomplete, his ingenious method of symmetrization enjoys a wide range of applications, see e.g. Bandle [5].

The Swedish mathematician Tage Gills Torsten Carleman (*1892 in Visseltofta, †1949 in Stockholm) was the first who proved the isoperimetric inequality

$$\mathscr{A}[X] \le \frac{\mathscr{L}[\Gamma]^2}{4\pi}$$

for minimal surfaces. Beckenbach and Radó in [9] then established this inequality on simply connected surfaces with non-positive Gaussian curvature.

The isoperimetric inequality encodes regularity properties of the underlying surface. Namely due to Federer and Fleming [54] and Maz'ya [118] it is equivalent to the Sobolev inequality

$$\left(\iint_B |\nabla\varphi|^2\,dudv\right) \ge 4\pi \iint_B \varphi^2\,dudv.$$

Finally Topping in [158] derived the isoperimetric inequality in the plane and on curved surfaces using results from the theory of curve shortening flow.

For further references and applications of the isoperimetric inequality we want to refer again to Osserman [129] and the references therein; see also the monographs Bandle [5] and Chavel [25].

14.7 The spherical energy of μ-stable geodesic discs

Let us finally come back to the stability condition

$$\iint_B \nabla_{ds^2}(\psi,\psi)W\,dudv \ge \mu \iint_B (q-K)W\psi^2\,dudv - 2\alpha(n)\mathscr{T}_X[N] \qquad (14.25)$$

used already in section 14.2 above.

Following methods from Sauvigny [141] we want to show how to apply this inequality to establish an upper bound for the spherical energy of a surface.

Theorem 14.6. *Let the conformally parametriced immersion $X: B \to \mathbb{R}^{n+2}$ together with an ONF N be given. Assume that X is stable in the sense of (14.25) with a function q satisfying*

$$H^2 - q \ge 0 \quad \text{in } B.$$

Then the Dirichlet energy of $N=(N_1,\ldots,N_n)$ *can be estimated as follows*

$$\iint\limits_{B_\nu}|\nabla N|^2\,dudv\le 2\iint\limits_{B_\nu}(2H^2-q)W\,dudv+\frac{8}{\mu\nu^2}+\left(2+\frac{2\alpha(n)}{\mu}\right)\mathscr{T}_X[N] \quad (14.26)$$

for all $\nu\in(0,1)$ *with the setting*

$$|\nabla N|^2:=\sum_{\sigma=1}^n|\nabla N_\sigma|^2\,.$$

Proof. For $\nu\in(0,1)$ we take the special test function $\Phi\in C_0^\infty(B,\mathbb{R})$ such that

$$\Phi\equiv 1\quad\text{on } B_\nu\,,\quad |\nabla\Phi|\le\frac{2}{\nu}\quad\text{on } B.$$

From the Weingarten equations we know

$$N_{\sigma,u}^2=\frac{L_{\sigma,11}^2+L_{\sigma,12}^2}{W}+\sum_{\vartheta=1}^n(T_{\sigma,1}^\vartheta)^2\,,\quad N_{\sigma,v}^2=\frac{L_{\sigma,12}^2+L_{\sigma,22}^2}{W}+\sum_{\vartheta=1}^n(T_{\sigma,2}^\vartheta)^2$$

using conformal parameters. Then we compute

$$\begin{aligned}
&\sum_{\sigma=1}^n\iint\limits_{B_\nu}|\nabla N_\sigma|^2\,dudv\\
&\quad=2\sum_{\sigma=1}^n\iint\limits_{B_\nu}(2H_\sigma^2-K_\sigma)W\,dudv+\sum_{\sigma,\vartheta=1}^n\iint\limits_{B_\nu}\Big\{(T_{\sigma,1}^\vartheta)^2+(T_{\sigma,2}^\vartheta)^2\Big\}\,dudv\\
&\quad=2\iint\limits_{B_\nu}(2H^2-q)W\,dudv+2\iint\limits_{B_\nu}(q-K)W\,dudv+2\mathscr{T}_X[N]\\
&\quad\le 2\iint\limits_{B_\nu}(2H^2-q)W\,dudv+2\iint\limits_{B}(q-K)W\Phi^2\,dudv+2\mathscr{T}_X[N]\\
&\quad\le 2\iint\limits_{B_\nu}(2H^2-q)W\,dudv+\frac{2}{\mu}\iint\limits_{B}|\nabla\Phi|^2\,dudv+2\left(1+\frac{2\alpha(n)}{\mu}\right)\mathscr{T}_X[N]\\
&\quad\le 2\iint\limits_{B_\nu}(2H^2-q)W\,dudv+\frac{8}{\mu\nu^2}+2\left(1+\frac{2\alpha(n)}{\mu}\right)\mathscr{T}_X[N]
\end{aligned}$$

prooving the statement. □

Chapter 15
F-minimal surfaces in $\mathbb{R}^3$

In this final chapter we consider general elliptic functionals and its critical points in Euclidean space $\mathbb{R}^3$. The Lagrangian densities of these functionals depend on the surface vector X as well as the normal direction $X_u \times X_v$.

15.1 F-minimal surfaces

This final chapter is devoted to analytical studies of critical points $X\colon B \to \mathbb{R}^3$ of anisotropic variational problems of the form

$$\mathscr{F}[X] = \iint\limits_B F(X_u \times X_v)\,dudv \longrightarrow \text{extr!}$$

Such critical points are called *F-minimal surfaces.* As we have seen in chapter 11, after introducing the special weight matrix

$$\mathbf{W}(Z)^{-1} = \frac{1}{\sqrt{\det \mathbf{F}_{ZZ}(Z)}}\,\mathbf{F}_{ZZ}(Z) + (z^i z^j)_{i,j=1,2,3}$$

those immersions are revealed as *weighted minimal surfaces.*

15.2 The geometry of the spherical mapping

On the Hopf function

From Theorem 5.4 we immediately obtain

Theorem 15.1. *Let the weighted conformally parametrized F-minimal surface $X\colon B \to \mathbb{R}^3$ be given. Then its Hopf function*

$$\mathscr{H}(w) = L_{11}(w) - L_{22}(w) - 2iL_{12}(w)$$

satisfies the complex differential equation

$$\mathscr{H}_{\overline{w}}(w) = A(w)\mathscr{H}(w) + B(w)\overline{\mathscr{H}}(w) \quad \text{in } B. \tag{15.1}$$

Here we set

$$A = \frac{1}{4}(a - d + ic + ib), \quad B = \frac{1}{4}(a + d + ic - ib)$$

as well as

$$a = \Omega^1_{22} + \Omega^2_{21}, \quad b = \Omega^2_{22} - \Omega^1_{21}, \quad c = \Omega^1_{12} + \Omega^2_{11}, \quad d = \Omega^2_{12} - \Omega^1_{11}.$$

Thus the spherical mapping $N = N(u,v)$ of the immersion possesses at most isolated interior branch points, or it holds $K \equiv 0$ at all.

Here we recall the identity

$$N_u \times N_v = KWN$$

with the Gaussian curvature K.

Analogously we would proceed in case of *prescribed non-weighted mean curvature.* Then the Hopf function is holomorphic and solves the Cauchy-Riemann equation

$$\mathscr{H}_{\overline{w}}(w) = 0 \quad \text{in } B.$$

The spherical mapping N thus possesses at most isolated interior branch points, or it holds $h_0^2 - K \equiv 0$ due to $|\mathscr{H}|^2 = 4(h_0^2 - K)W^2$. We refer e.g. to Jost [101].

An asymptotical expansion

Our next result on the distribution of the branch points of the sphercial mapping goes back Sauvigny [141].

Theorem 15.2. *Let the weighted conformally parametrized F-minimal surface $X\colon B \longrightarrow \mathbb{R}^3$ be given. Then at each point $w_0 \in B$ there are two linearly independent vectors $A, B \in \mathbb{R}^3$ and a natural number $n \in \mathbb{N}$ such that the following asymptotical expansion holds true*

$$N(w_0 + w) - N(w_0) = \frac{2}{n}\rho^n(A\cos n\varphi + B\sin n\varphi) + o(\rho^n) \quad \textit{for } \rho \longrightarrow 0. \qquad (15.2)$$

Proof. From (11.33) we already know

$$|\Delta N| \le (1 + \omega_2)|\nabla N|^2 \quad \text{in } B.$$

Thus due to a theorem of Hartman and Wintner [77] (see also Hildebrandt [84]) at each point $w_0 \in B$ there is a vector $C := A - iB$, $C \neq 0$, and a number $n \in \mathbb{N}$ such that it holds

$$N_w(w_0 + w) = \frac{1}{2}\Big\{N_u(w_0 + w) - iN_v(w_0 + w)\Big\} = Cw^{n-1} + o(|w|^{n-1}) \qquad (15.3)$$

for $w \longrightarrow 0$.

The linear connection between the weighted fundamental forms from section 5.1 implies that $I\!I\!I_W(X)$ und $I_W(X)$ are diagonalized simultaneously. We get

$$0 = \left\{ \mathbf{W}^{-\frac{1}{2}}(N) \circ N_w \right\}^2 = \left\{ \left(\mathbf{W}^{-\frac{1}{2}}(N) \circ C \right)^2 + o(1) \right\} w^{2n-2} \quad \text{for } w \to 0.$$

For the non-singular matrix $\mathbf{F} := \mathbf{W}^{-\frac{1}{2}}(N(w_0))$ this means

$$0 = (\mathbf{F} \circ C)^2 = |\mathbf{F} \circ A|^2 - |\mathbf{F} \circ B|^2 - 2i(\mathbf{F} \circ A) \cdot (\mathbf{F} \circ B),$$

and we conclude

$$(\mathbf{F} \circ A) \cdot (\mathbf{F} \circ B) = 0, \quad |\mathbf{F} \circ A| = |\mathbf{F} \circ B| > 0. \tag{15.4}$$

Thus A and B are linearly independent. We introduce polar coordinates and integrate (15.3) to get

$$\begin{aligned} N(w_0 + w) - N(w_0) &= \frac{1}{n} (Cw^n + \overline{C}\overline{w}^n) + o(|w|^n) \\ &= \frac{2}{n} \operatorname{Re} \left\{ (A - iB)\rho^n (\cos n\varphi + i \sin n\varphi) \right\} + o(\rho^n) \\ &= \frac{2}{n} \rho^n (A \cos n\varphi + B \sin n\varphi) + o(\rho^n), \qquad \rho \to 0. \end{aligned}$$

The statement is proved. □

In other words: *The spherical mapping is open, an open neighbourhood of a point $w \in \mathring{B}$ is mapped onto an open neighbourhood of its image $N(w) \in S^2$.*

We say $w \in B$ is a *branch point of order $n - 1$* if the expansion (15.2) holds true with $n \in \mathbb{N}$.

The spherical mapping under the stereographic projection

Shortly we want to prove *quasiconformality of the spherical mapping of weighted minimal surfaces.* For this purpose we start with some basic facts about the stereographic projection from $S^2 \subset \mathbb{R}^3$ into $\mathbb{R}^2$.

Let an angle $\omega \in (0, \pi)$ be given, and suppose that

$$N(u,v) \cdot (0,0,1) \geq \cos\omega \quad \text{in } B, \quad \omega \in [0, \pi), \tag{15.5}$$

holds true for the spherical image of the surface or equivalently

$$N(u,v) \in S^2_\omega \quad \text{in } B$$

with the spherical cap

$$S^2_\omega := \Big\{ Z \in S^2 \,:\, Z \cdot (0,0,1) \ge \cos\omega \Big\} \subset S^2 .$$

The Gauss map is supposed to skip a neighborhood of the south pole.

Now we parametrize S^2_ω as follows

$$\begin{aligned} &z_1 \,=\, \sin\vartheta\cos\varphi, \quad z_2 \,=\, \sin\vartheta\sin\varphi, \quad z_3 \,=\, \cos\vartheta, \\ &\vartheta \in [0,\omega],\ \varphi \in [0,2\pi], \end{aligned} \tag{15.6}$$

and by

$$\mathrm{Sp}\colon S^2_\omega \longrightarrow \mathbb{R}^2$$

we denote the stereographic projection with the south pole as center, analytically given by (see e.g. Neutsch [125])

$$\mathrm{Sp}(z_1,z_2,z_3) = \xi + i\eta \equiv \frac{z^1}{1+z^3} + i\,\frac{z^2}{1+z^3}\,, \quad (z_1,z_2,z_3) \in S^2_\omega\,. \tag{15.7}$$

Its inverse suffices

$$z^1 = \frac{2\xi}{1+\xi^2+\eta^2}\,, \quad z^2 = \frac{2\eta}{1+\xi^2+\eta^2}\,, \quad z^3 = \frac{1-\xi^2-\eta^2}{1+\xi^2+\eta^2}\,.$$

Using the spherical coordinates from (15.6) we can write

$$\mathrm{Sp}(\vartheta,\varphi) = \left(\frac{\sin\vartheta\cos\varphi}{1+\cos\vartheta}\,, \frac{\sin\vartheta\sin\varphi}{1+\cos\vartheta} \right), \tag{15.8}$$

and together with (15.7) we calculate

$$1+\xi^2+\eta^2 = 1 + \frac{\sin^2\vartheta\cos^2\varphi}{(1+\cos\vartheta)^2} + \frac{\sin^2\vartheta\sin^2\varphi}{(1+\cos\vartheta)^2} = \frac{2}{1+\cos\vartheta}\,.$$

In this way we are led to the so-called *modulus of distortion*

$$\sigma(\vartheta,\varphi) := \frac{1}{1+\cos\vartheta}\,. \tag{15.9}$$

For example, the north pole is distorted by the factor $\frac{1}{2}$, while along the equator all lengths remain fixed.

Quasiconformality of the spherical mapping

Our next results states quasiconformality of the spherical mapping under the conditions previously discussed. We follow again Sauvigny [141].

Proposition 15.1. *Let the weighted conformally parametrized weighted minimal surface $X\colon B \to \mathbb{R}^3$ be given. Assume that its spherical mapping satisfies (15.5) with an angle $\omega \in [0,\pi)$. Then the associated plane mapping*

$$\Phi(u,v) := \mathrm{Sp} \circ N(u,v), \quad (u,v) \in B,$$

is bounded and quasiconform, i.e. there hold

$$|\Phi(u,v)| \le \frac{\sin\omega}{1+\cos\omega} \quad \textit{and} \quad |\nabla\Phi(u,v)|^2 \le -2(1+\omega_0)J_\Phi(u,v) \quad \textit{in } B$$

with the non-positive Jacobian $J_\Phi = J_\Phi(u,v)$ of the mapping $\Phi = \Phi(u,v)$.

Proof. The first bound follows directly from the representation (15.8) of the stereographic projection in terms of ϑ and φ. Furthermore, together with (15.9) we estimate the distortion $\chi = \chi(\xi,\eta)$ of the inverse mapping Sp^{-1}, i.e.

$$1+\cos\omega \le \chi(\xi,\eta) \le 2 \quad \text{if } \mathrm{Sp}^{-1}(\xi,\eta)\cdot(0,0,1) \ge \cos\omega. \tag{15.10}$$

Now we define the weight matrix

$$\mathbf{H}(u,v) := \frac{1}{\chi(\Phi(u,v))^2}\,\partial\mathrm{Sp}^{-1}(\Phi(u,v))^T \circ \mathbf{W}(X,N)^{-1} \circ \partial\mathrm{Sp}^{-1}(\Phi(u,v))$$

satisfying

$$(1+\omega_0)^{-1}|\xi|^2 \le \xi \circ \mathbf{H}(u,v) \circ \xi \le (1+\omega_0)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2\ (u,v) \in B$$

as well as

$$\det\mathbf{H}(u,v) = 1 \quad \text{in } B.$$

Using the linear connection between the weighted fundamental forms we infer

$$\partial\Phi(u,v)^T \circ \mathbf{H}(u,v) \circ \partial\Phi(u,v) = -WK\chi^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice the diagonal form of this product. Since $K \le 0$ we have

$$\begin{aligned} -\det\partial\Phi &= -\det\left\{\partial\Phi^T \circ \mathbf{H} \circ \partial\Phi\right\}^{\frac{1}{2}} = -WK\chi^2 = \frac{1}{2}\,\mathrm{Spur}\left\{\partial\Phi^t \circ \mathbf{H} \circ \partial\Phi\right\} \\ &\ge \frac{1}{2(1+\omega_0)}\,|\nabla\Phi|^2. \end{aligned}$$

This proves the statement. □

Quasiconformality of the spherical mapping for immersions of this surface class along with curvature estimates were already established in Simon [147] in 1977, see also our following considerations. We additionally want to refer the reader to Hoy [95], [96] and [97] on this subject.

An oscillation estimate for the spherical mapping

Now Theorem 12.3 from Gilbarg and Trudinger [71] (see also Simon [147]) provides an interior Hölder estimate for so-called (k,k')-quasiconformal plane mappings $(p,q) = (p(x,y), q(x,y))$ in the sense of

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \le 2k(p_x q_y - p_y q_x) + k'$$

with constants $|k| \ge 1$ and $k' \ge 0$. Together with our modulus of distortion (15.10) we arrive at (see Sauvigny [141])

Theorem 15.3. *Let the weighted conformally parametrized F-minimal surface $X\colon B \to \mathbb{R}^3$ be given. Suppose that its spherical mapping satisfied (15.5) with an angle $\omega \in [0,\pi)$. Let finally $\lambda > 0$ and $\nu \in (0,1)$ be fixed. Then there exists $\delta = \delta(\omega_0, \omega, \nu; \lambda) \in (0, \frac{1}{2}\nu)$ such that for all $w_0 \in B_{1-\frac{1}{2}\nu}(0,0)$ the following oscillation estimate holds true*

$$|N(w) - N(w_1)| \le \lambda \quad \textit{for all } w \in B_\delta(w_0) \tag{15.11}$$

with arbitrary $w_1 \in B_\delta(w_0)$.

Application to immersion of prescribed weighted mean curvature

Let us consider more closely the spherical mapping of immersions with prescibed mean curvature from the point of view of quasiconformality.

Due to the linear connection

$$I\!I\!I_W(X) - 2H_W(X,N)I\!I_W(X) + KI_W(X) = \mathbf{0}$$

between the weighted fundamental forms we compute

$$(\det \partial\Phi)^2 = \det \begin{pmatrix} 2H_W(X,N)L_{11} - KW & 2H_W(X,N)L_{12} \\ 2H_W(X,N)L_{21} & 2H_W(X,N)L_{22} - KW \end{pmatrix} \chi^2 = K^2W^2\chi^4$$

using weighted conformal parameters $(u,v) \in B$. i.e. for the signed Jacobian $\partial\Phi$ we infer the identity

$$\det \partial\Phi = KW\chi^2\,.$$

Moreover, for its trace we have

$$\begin{aligned} \operatorname{trace}(\partial\Phi^T \circ \mathbf{H} \circ \partial\Phi) &= \Big\{2H_W(X,N)L_{11} - KW + 2H_W(X,N)L_{22} - KW\Big\}\chi^2 \\ &= 2\Big\{2H_W(X,N)^2 - K\Big\}W\chi^2\,. \end{aligned}$$

Proposition 15.2. *Using weighted conformal parameters* $(u,v) \in B$ *it holds*

$$|\nabla \Phi|^2 \le -2(1+\omega_0)\det \partial \Phi + 4(1+\omega_0)H_W G(X,N)^2 W \chi^2 \quad \textit{in } B,$$

i.e. $\Phi = \Phi(u,v)$ *is* (k,k')*-quasiconformal with*

$$k = -2(1+\omega_0) \quad \textit{and} \quad k' = 4(1+\omega_0)H_W(X,N)^2 W \chi^2\,.$$

We want to apply this reslut to immersions with *bounded gradient.* Suppose

$$\sup_{(u,v)\in B} W(u,v) \le W_0 \in (0,+\infty). \tag{15.12}$$

Then we find

Theorem 15.4. *Assume that the weighted conformally parametrized immersion* $X\colon B \to \mathbb{R}^3$ *satisfies (15.12) and that its spherical mapping fulfills (15.5) with an angle* $\omega \in [0,\pi)$*. Let finally* $\lambda > 0$ *and* $\nu \in (0,1)$ *be fixed. Then there exist* $\delta = \delta(\omega_0,\omega,\nu,W_0;\lambda) \in (0,\frac{1}{2}\nu)$ *such that for all* $w_0 \in B_{1-\frac{1}{2}\nu}(0,0)$ *the following oscillation estimate holds true*

$$|N(w) - N(w_1)| \le \lambda \quad \textit{for all } w \in B_\delta(w_0) \tag{15.13}$$

with arbitrary $w_1 \in B_\delta(w_0)$.

This estimate can immediately be applied to conformally parametrized immersions with prescribed non-weighted mean curvature $H(X,Z)$ *which are small in the sense of*

$$h_0 M < 1 \quad \text{with } M := \sup_{(u,v)\in B} |X(u,v)|, \quad h_0 := \sup_{(X,Z)\in\mathbb{R}^3\times\mathbb{R}^3} |H(X,Z)|.$$

This follows from Heinz' gradient estimates from section 9.4 taking

$$|\Delta X| \le h_0 |\nabla X|^2 \quad \text{in } B$$

into account. Thus there is a real constant $c = c(h_0,M,\nu_1)$ with the property

$$|\nabla X(u,v)| \le c(h_0,M,\nu_1) \quad \text{for all } (u,v) \in B_{1-\nu_1}(0,0).$$

Then the mapping $N = N(u,v)$ *is* (k,k')*-quasiconformal in* $B_{1-\nu_1}(0,0)$ *with* $k = -2$ *and* $k' = 4h_0^2 c(h_0,M,\nu_1)^2$.

Furthermore, the smallness condition can be replaced by the following modulus of continuity:

For real $\lambda > 0$ *there is a* $\delta = \delta(\lambda)$ *such that*

$$|X(w) - X(w_1)| \le \lambda \quad \text{for all } w \in B_\delta(w_0) \tag{15.14}$$

for arbitrary $w_1 \in B_\delta(w_0)$ *and* $w_0 \in B$ *with* $B_\delta(w_0) \subset\subset B$.

Namely, define the mapping

$$\widehat{X}(w) := \frac{1}{\lambda}\left\{X(w) - X(w_1)\right\} : B_\delta(w_0) \longrightarrow \mathbb{R}^3$$

with the properties

$$|\widehat{X}(w)| \le 1 \quad \text{and} \quad |\Delta\widehat{X}(w)| \le \lambda h_0 |\nabla\widehat{X}(w)|^2 \quad \text{in } B_\delta(w_0).$$

Now choose $\lambda > 0$ so that $\lambda h_0 < 1$. This implies an inner gradient estimate for the surface vector and therefore an oscillation estimate for spherical mapping.

Finally we want to remark that an oscillation estimates for the composition

$$N(u,v) = \mathfrak{G} \circ X(u,v), \quad (u,v) \in B,$$

with the Gauss map $\mathfrak{G}\colon X(B) \to S^2$ *on surface graphs of mean curvature type* is established in Gilbarg and Trudinger [71], section 16.6. Assuming a modulus of continuity (15.14) we could then infer an oscillation estimate $N = N(u,v)$.

15.3 Stability and μ-stability

The second variation

Our aim now is to compute the second variation of the parametric functional $\mathscr{F}[X]$.

With a test function $\varphi \in C_0^\infty(B,\mathbb{R})$ we consider the normal variation

$$\widetilde{X}(u,v) := X(u,v) + \varepsilon\varphi(u,v)N(u,v), \quad (u,v) \in B.$$

Then it follows

$$\begin{aligned}\widetilde{X}_u \wedge \widetilde{X}_v = {} & X_u \wedge X_v + \varepsilon\{X_u \wedge N_v + N_u \wedge X_v\}\varphi + \varepsilon\{X_u \wedge N\varphi_v + N \wedge X_v\varphi_u\} \\ & + \varepsilon^2\varphi^2 N_u \wedge N_v + \varepsilon^2\{N_u \wedge N\varphi_v + N \wedge N_v\varphi_u\}\varphi.\end{aligned}$$

First it holds

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon^2} F(\widetilde{X}_u \wedge \widetilde{X}_v)\Big|_{\varepsilon=0} = {} & \frac{\partial}{\partial\varepsilon}(\widetilde{X}_u \wedge \widetilde{X}_v)\Big|_{\varepsilon=0} \circ \mathbf{F}_{ZZ}(X_u \wedge X_v) \circ \frac{\partial}{\partial\varepsilon}(\widetilde{X}_u \wedge \widetilde{X}_v)\Big|_{\varepsilon=0} \\ & + F_Z(X_u \wedge X_v) \cdot \frac{\partial^2}{\partial\varepsilon^2}(\widetilde{X}_u \wedge \widetilde{X}_v)\Big|_{\varepsilon=0}.\end{aligned}$$

Let $(u,v) \in B$ be weighted conformal parameters. From the properties of the weight matrix $\mathbf{W}(Z)$ we infer

$$
\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon^2} F(\widetilde{X}_u \wedge \widetilde{X}_v)\Big|_{\varepsilon=0} \\
&= \frac{1}{W}\Big\{(X_u \wedge N_v + N_u \wedge X_v)\varphi + (X_u \wedge N\varphi_v + N \wedge X_v\varphi_u)\Big\} \circ \mathbf{F}_{ZZ}(N) \circ \ldots \\
&\qquad \ldots \circ \Big\{(X_u \wedge N_v + N_u \wedge X_v)\varphi + (X_u \wedge N\varphi_v + N \wedge X_v\varphi_u)\Big\} \\
&\qquad + F_Z(N) \cdot \Big\{2N_u \wedge N_v\varphi^2 + N_u \wedge N(\varphi^2)_v + N \wedge N_v(\varphi^2)_u\Big\} \\
&= \frac{\sqrt{\det \mathbf{F}_{ZZ}(N)}}{W}(X_u \wedge N\varphi_v + N \wedge X_v\varphi_u) \circ \mathbf{W}(N)^{-1} \circ (X_u \wedge N\varphi_v + N \wedge X_v\varphi_u) \\
&\qquad + \Big\{\big[F_Z(N), N_u, N\big]\varphi^2\Big\}_v + \big\{\big[F_Z(N), N, N_v\big]\varphi^2\big\}_u \\
&\qquad - \big[\mathbf{F}_{ZZ}(N) \circ N_v, N_u, N\big]\varphi^2 - \big[\mathbf{F}_{ZZ}(N) \circ N_u, N, N_v\big]\varphi^2
\end{aligned}
$$

with the usual setting $[X,Y,Z] := X \cdot (Y \times Z)$. It follows that

$$
\begin{aligned}
&(X_u \wedge N\varphi_v + N \wedge X_v\varphi_u) \circ \mathbf{W}(N)^{-1} \circ (X_u \wedge N\varphi_v + N \wedge X_v\varphi_u) \\
&= \Big\{\mathbf{W}(N)^{\frac{1}{2}} \circ X_v\varphi_v + \mathbf{W}(N)^{\frac{1}{2}} \circ X_u\varphi_u\Big\} \circ \Big\{\mathbf{W}(N)^{\frac{1}{2}} \circ X_v\varphi_v + \mathbf{W}(N)^{\frac{1}{2}} \circ X_u\varphi_u\Big\} \\
&= W\Big\{\varphi_u^2 + \varphi_v^2\Big\} = W|\nabla\varphi|^2 .
\end{aligned}
$$

For the divergence term we have

$$
\iint\limits_B \operatorname{div}\Big(\big[F_Z(N), N, N_v\big]\varphi^2, \big[F_Z(N), N_u, N\big]\varphi^2\Big)\, du\, dv = 0
$$

due to $\varphi = 0$ on $\partial B = 0$. Finally we know

$$
\big[N_u, N_v, \mathbf{F}_{ZZ}(N) \circ N\big] = 0,
$$

so that we infer

$$
\begin{aligned}
&\big[\mathbf{F}_{ZZ}(N) \circ N_u, N_v, N\big] + \big[N_u, \mathbf{F}_{ZZ}(N) \circ N_v, N\big] \\
&\quad = \big[\mathbf{F}_{ZZ}(N) \circ N_u, N_v, N\big] + \big[N_u, \mathbf{F}_{ZZ}(N) \circ N_v, N\big] + \big[N_u, N_v, \mathbf{F}_{ZZ}(N) \circ N\big] \\
&\quad = \big[N_u, N_v, N\big] \operatorname{trace} \mathbf{F}_{ZZ}(N) = KW \operatorname{trace} \mathbf{F}_{ZZ}(N).
\end{aligned}
$$

Thus we arrive at

$$\delta^2\mathscr{F}[X] = \iint\limits_B \left\{ \sqrt{\det \mathbf{F}_{ZZ}(N)}\, |\nabla\varphi|^2 + KW \operatorname{trace} \mathbf{F}_{ZZ}(N)\varphi^2 \right\} dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$.

Using general parameters $(u,v) \in B$ we have therefore proved

Theorem 15.5. *The second variation of the parametric functional $\mathscr{F}[X]$ reads*

$$\delta^2\mathscr{W}[X] = \iint\limits_B \left\{ \sqrt{\det \mathbf{F}_{ZZ}(N)}\, \nabla_{ds_W^2}(\varphi,\varphi) + K \operatorname{trace} \mathbf{F}_{ZZ}(N)\varphi^2 \right\} W\, dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$.

Stability and μ-stability

Again we require

$$M_1|\xi|^2 \le \xi \circ \mathbf{F}_{ZZ}(Z) \circ \xi \le M_2|\xi|^2 \quad \text{for all} \quad \xi = (\xi^1,\xi^2,\xi^3) \in \mathbb{R}^3 \tag{15.15}$$

and all $Z \in S^2$ with $\xi \cdot Z = 0$, with positive constants $0 < M_1 \le M_2 < +\infty$. Then we can set

$$1+\omega_0 = \frac{M_2}{M_1} \quad \text{resp.} \quad \frac{1}{1+\omega_0} = \frac{M_1}{M_2}\,.$$

Theorem 15.6. *Let $X\colon B \to \mathbb{R}^3$ be a stable F-minimal surface, i.e. assume that*

$$\delta^2\mathscr{F}[X] \ge 0.$$

Furthermore, suppose that (15.15) is satisfied. Let finally $\mu > 0$ be a real number with the propery

$$\mu \le \frac{2M_1}{M_2}\,.$$

Then the immersion is μ-stable with this number μ, i.e. it holds

$$\iint\limits_B \nabla^2_{ds_W}(\varphi,\varphi)W\, dudv \ge \mu \iint\limits_B (-K)W\varphi^2\, dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$.

Proof. Namely after rearranging it follows

$$M_2 \iint\limits_B \nabla_{ds_W^2}(\varphi,\varphi)W\, dudv \ge 2M_1 \iint\limits_B (-K)W\varphi^2\, dudv$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$.

This implies

$$\iint\limits_B \nabla_{ds_W^2}(\varphi,\varphi)W\,dudv \geq \frac{2M_1}{M_2}\iint\limits_B (-K)W\varphi^2\,dudv,$$

and the statement follows. □

15.4 Eigenvalue problems for the spherical mapping

This section is devoted to a deeper analysis of μ-stable F-minimal surfaces in Euclidean space $\mathbb{R}^3$. Our results are partly build up on our considerations from chapter 13, on the other hand we will meet new methods and results applicable for F-minimal surfaces in $\mathbb{R}^3$ exclusively.

A comparison principle

The spherical mapping $N\colon\, B\to\mathbb{R}^3$ represents a branched immersion from the closed unit disc $B\subset\mathbb{R}^2$ into the unit sphere $S^2\subset\mathbb{R}^3$. As we have seen above, interior branch points are isolated.

In the following sections we are concerned with the eigenvalue problem 13.6 on spherical domains. To establish suitable bounds for the corresponding first eigenvalue $\lambda_1>0$ we need the following comparison argument.

Lemma 15.1. *Let the immersion $X\colon\, B\to\mathbb{R}^3$ be given. Let*

$$ds^2=\sum_{i,j=1}^2 g_{ij}\,du^idu^j\,,\quad ds_W^2=\sum_{i,j=1}^2 h_{ij}\,du^idu^j$$

its non-weighted resp. weighted line elements w.r.t. to a weight matrix $\mathbf{W}(Z)$. Then there hold

$$\begin{aligned}(1+\omega_0)^{-1}\iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,dudv &\leq \iint\limits_B \nabla_{ds_W^2}(\varphi,\varphi)W\,dudv\\ &\leq (1+\omega_0)\iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,dudv\end{aligned} \tag{15.16}$$

for all $\varphi\in C_0^1(B,\mathbb{R})$ with the invariant Beltrami operators ∇_{ds^2} and $\nabla_{ds_W^2}$.

Proof. On the tangential plane $\mathfrak{T}_X$ of the immersion we compute

$$\begin{aligned}
&\iint\limits_B \nabla_{ds_W^2}(\varphi,\varphi)W\,dudv \\
&\quad= \iint\limits_B (\varphi_u,\varphi_v)\circ\left\{\partial X^T\circ\mathbf{W}(X,N)\circ\partial X\right\}^{-1}\circ(\varphi_u,\varphi_v)W\,dudv \\
&\quad= \iint\limits_B (\varphi_u,\varphi_v)\circ(\partial X)^{-1}\circ\mathbf{W}(X,N)^{-1}\circ(\partial X^T)^{-1}\circ(\varphi_u,\varphi_v)W\,dudv \\
&\quad= \iint\limits_B \left\{(\varphi_u,\varphi_v)\circ(\partial X)^{-1}\right\}\circ\mathbf{W}(X,N)^{-1}\circ\left\{(\varphi_u,\varphi_v)\circ(\partial X)^{-1}\right\}W\,dudv
\end{aligned}$$

from where we infer the estimates

$$\begin{aligned}
&\iint\limits_B \nabla_{ds_W^2}(\varphi,\varphi)W\,dudv \\
&\quad\le (1+\omega_0)\iint\limits_B \left\{(\varphi_u,\varphi_v)\circ(\partial X)^{-1}\right\}\cdot\left\{(\varphi_u,\varphi_v)\circ(\partial X)^{-1}\right\}W\,dudv \\
&\quad= (1+\omega_0)\iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,dudv
\end{aligned}$$

as well as

$$\begin{aligned}
&\iint\limits_B \nabla_{ds_W^2}(\varphi,\varphi)W\,dudv \\
&\quad\ge \frac{1}{1+\omega_0}\iint\limits_B \left\{(\varphi_u,\varphi_v)\circ(\partial X)^{-1}\right\}\cdot\left\{(\varphi_u,\varphi_v)\circ(\partial X)^{-1}\right\}W\,dudv \\
&\quad= \frac{1}{1+\omega_0}\iint\limits_B \nabla_{ds^2}(\varphi,\varphi)W\,dudv
\end{aligned}$$

proving the statement. □

The weighted Rayleigh quotient on S^2

Let $\Omega\subset S^2$ be a spherical domain covered by the Gauss map $N=N(u,v)$ of the F-minimal surface $X\colon B\to\mathbb{R}^3$.

To the Rayleigh quotient $\mathscr{R}[\varphi]$ introduced in section 13.5 we define its weighted counterpart

$$\mathscr{R}_W[\varphi] := \frac{\int\limits_\Omega \nabla_{S^2,W}(\varphi,\varphi)\,dS}{\int\limits_\Omega \varphi^2\,dS}$$

for funktions $\varphi \in V(\Omega,\mathbb{R})$ on the function space

$$V(\Omega,\mathbb{R}) := \left\{\phi \in H^{1,2}(\Omega,\mathbb{R}) \cap C^0(B,\mathbb{R}) : \phi \not\equiv 0,\ \phi = 0 \text{ on } \partial\Omega\right\}$$

with the weighted Beltrami operator $\nabla_{S^2,W}$ on $S^2 \subset \mathbb{R}^3$ and its area element dS.

To $\Omega \subset S^2$ we associate the first eigenvalue of this weighted operator on S^2 :

$$\min_{\varphi\in V(\Omega,\mathbb{R})} \mathscr{R}_W[\varphi] = \lambda_{1,W}(\Omega).$$

From the previous comparison result we conclude

$$\frac{1}{1+\omega_0}\lambda_{1,W}(\Omega) \le \lambda_1(\Omega) \le (1+\omega_0)\lambda_{1,W}(\Omega). \tag{15.17}$$

μ-stability due to Schwarz

The following result characterizes the size of the spherical image of μ-stable F-minimal surfaces, see Fröhlich [62].

Theorem 15.7. *Let the weighted conformally parametrized and μ-stable F-minimal surface $X : B \to \mathbb{R}^3$ with $\mu > 0$ be given. Let $\Omega' \subset\subset \Omega \subset S^2$ denote the domain which is covered by the spherical mapping $N|_{\mathring{B}'}$ on an open disc $\mathring{B}' \subset\subset \mathring{B}$. Furthermore assume that $N(\partial B') = \partial\Omega'$. Then for the first eigenvalue $\lambda_1(\Omega')$ of the spherical Laplacian on $\Omega' \subset S^2$ it holds*

$$\lambda_1(\Omega') \ge \frac{\mu}{1+\omega_0}\,. \tag{15.18}$$

Proof. For $\lambda_{1,W}(\Omega')$ we choose an eigenfunction $\varphi : \overline{\Omega'} \to \mathbb{R}$ such that

$$\begin{aligned} \Delta_{S^2,W}\varphi + \lambda_{1,W}(\Omega')\varphi &= 0 \quad \text{in } \Omega',\\ \varphi &= 0 \quad \text{auf } \partial\Omega',\\ \varphi &> 0 \quad \text{in } \Omega'. \end{aligned}$$

The finitely many branch points of the Gauss map on the closed set $\overline{B'} \subset \mathring{B}$ are denoted by $\{w_1,\ldots,w_N\}$ with $N \in \mathbb{N}$. Now for sufficiently small $\varepsilon > 0$ we consider closed discs $K_\varepsilon(w_t) \subset \overline{B'}$ (eventually together with their intersection with $\overline{B'}$) of common radius $\varepsilon > 0$ around these branch points. The pull-back $\psi(u,v) := \varphi \circ N(u,v)$, $(u,v) \in \overline{B'}$, then satisfies the eigenvalue problem

$$\begin{aligned} \Delta_{ds_g^2}\psi - \lambda_{1,W}(\Omega')K\psi = 0 \quad & \text{in } B'_\varepsilon := B' \setminus \bigcup_{t=1}^{N} K_\varepsilon(w_t), \\ \psi = 0 \quad & \text{on } \partial B', \\ \psi > 0 \quad & \text{in } B'. \end{aligned}$$

Considering the vanishing boundary data of ψ we evaluate the μ-stability condition as follows

$$\begin{aligned} &\iint\limits_{B'_\varepsilon} |\nabla\psi|^2\,dudv + \mu \iint\limits_{B'_\varepsilon} \psi^2 KW\,dudv \\ &\quad = -\iint\limits_{B'_\varepsilon} \psi \triangle\psi\,dudv + \mu \iint\limits_{B'_\varepsilon} \psi^2 KW\,dudv + \sum_{t=1}^{N} \int\limits_{\partial K_\varepsilon(w_t)} \psi\nabla\psi \cdot (N^1,N^2)\,ds \\ &\quad = \left\{\lambda_{1,W}(\Omega') - \mu\right\} \iint\limits_{B'_\varepsilon} \psi^2(-K)W\,dudv + \sum_{t=1}^{N} \int\limits_{\partial K_\varepsilon(w_t)} \psi\nabla\psi \cdot (N^1,N^2)\,ds \end{aligned}$$

with the outer unit normal vector (N^1,N^2) at the boundary curve $\partial K_\varepsilon(w_t)$, $t = 1,\ldots,N$, and its line element ds. Since $\psi \in H_1^2(\mathring{B}',\mathbb{R}) \cap C^0(B',\mathbb{R})$ we infer

$$\begin{aligned} &\lim_{\varepsilon\to 0} \iint\limits_{B'_\varepsilon} \psi^2(-K)W\,dudv = \iint\limits_{B'} \psi^2(-K)W\,dudv, \\ &\lim_{\varepsilon\to 0} \sum_{t=1}^{N} \int\limits_{\partial K_\varepsilon(w_t)} \psi\nabla\psi \cdot (N^1,N^2)\,ds = 0. \end{aligned}$$

Thus if $\lambda_{1,W}(\Omega') - \mu < 0$ then it follows necessarily

$$\iint\limits_{B'} |\nabla\psi|^2\,dudv + \mu \iint\limits_{B'} \psi^2 KW\,dudv < 0$$

contradicting the μ-stability. $\square$

Spherical oscillation of μ-stable F-minimal surfaces

This result, the ideas of which go already back to H.A. Schwarz, leads us to our next oscillation estimate for the spherical mapping, see Fröhlich [62].

Theorem 15.8. *Let the weighted conformally parametrized and μ-stable F-minimal surface $X\colon B\to\mathbb{R}^3$ with $\mu>0$ be given. Then for each $\varepsilon=\varepsilon(\omega_0,\mu)>0$ satisfying (15.20) there is $\delta=\delta(\omega_0,\mu,\nu;\varepsilon(\mu))\in\big(0,(1-\nu)^2\big)$ with the property*

$$\operatorname*{osc}_{\overline{B}_{\delta\rho}(w_0)} N(w) := \sup_{w_1,w_2\in\overline{B}_{\delta\rho}(w_0)} |N(w_1)-N(w_2)| \le 2\varepsilon(\omega_0,\mu) \tag{15.19}$$

on each open disc $\mathring{B}_\rho(w_0)\subset B_{1-\nu}(0,0)$ with fixed $\nu\in(0,1)$ and $|w_0|<1-\nu$.

Proof. Consider the open disc $\mathring{B}_\rho(w_0)\subset B_{1-\nu}(0,0)$ with $\rho\in(0,1-\nu-|w_0|)$ and $|w_0|<1-\nu$, $\nu\in(0,1)$. Due to the estimate

$$\iint\limits_{|w|\le 1-\nu} |\nabla N|^2\,dudv \le \frac{8\pi(1+\omega_0)}{\mu\nu^2}$$

we can apply the Courant-Lebesgue lemma from Heinz [80], Lemma 14, to find for given $\varepsilon>0$ a real $\delta=\delta(\omega_0,\mu,\nu;\varepsilon)\in\big(0,(1-\nu)^2\big)$ and $\delta^*\in[\delta,\sqrt{\delta}]$ such that

$$\int\limits_{\partial B_{\delta^*\rho}(w_0)} |dN| \le 2\sqrt{\frac{8\pi^2(1+\omega_0)}{(-\log\delta)\mu\nu^2}} \le 2\varepsilon;$$

see the proof of our curvature estimate from chapter 10. This means in other words

$$N\big(\partial B_{\delta^*\rho}(w_0)\big)\subset K_\varepsilon\big(N(w_0+\delta^*\rho)\big)$$

with the setting $K_\varepsilon(Z):=\{X\in\mathbb{R}^3\,:\,|X-Z|\le\varepsilon\}$. Now let $\Omega_{\delta^*\rho}\subset S^2$ be covered by the restriction $N|_{\mathring{B}_{\delta^*\rho}(w_0)}$. Then it holds with a uniquely determined angle $\omega=\omega(\varepsilon)$

$$N\big(\partial B_{\delta^*\rho}(w_0)\big)\subset S^2_{\omega(\varepsilon)}(w_0):=S^2\cap K_\varepsilon\big(N(w_0+\delta^*\rho)\big).$$

We choose $\varepsilon=\varepsilon(\omega_0,\mu)$ sufficiently small with the property

$$\lambda_1\big(S^2\setminus S^2_{\omega(\varepsilon)}(w_0)\big)<\frac{\mu}{1+\omega_0}\,. \tag{15.20}$$

But then it mus be satisfied

$$\overline{\Omega}_{\delta^*\rho}\subset S^2_{\omega(\varepsilon)}(w_0).$$

Otherwise there is a point $w_1\in B_{\delta^*\rho}(w_0)$ with

$$N(w_1)\in S^2\setminus S^2_{\omega(\varepsilon)}(w_0).$$

Note that $N=N(u,v)$ is an open mapping as we showed above. Continuation along paths we arrive at

$$S^2\setminus S^2_{\omega(\varepsilon)}(w_0)\subset\overline{\Omega}_{\delta^*\rho}\,.$$

To $\lambda_1\big(S^2\setminus S^2_{\omega(\varepsilon)}(w_0)\big)$ we now choose an eigenfunction $\varphi\colon S^2\setminus S^2_{\omega(\varepsilon)}(w_0)\to\mathbb{R}$ such that there hold

$$\begin{aligned}\Delta_{S^2}\varphi+\lambda_1\big(S^2\setminus S^2_{\omega(\varepsilon)}(w_0)\big)\varphi&=0 \quad \text{in } S^2\setminus S^2_{\omega(\varepsilon)}(w_0),\\ \varphi&=0 \quad \text{on } \partial\big(S^2\setminus S^2_{\omega(\varepsilon)}(w_0)\big),\\ \varphi&>0 \quad \text{in } S^2\setminus S^2_{\omega(\varepsilon)}(w_0),\end{aligned}$$

which we pull back by means of $\psi(u,v):=\varphi\circ N(u,v)$, $(u,v)\in B_{\delta^*\rho}$. But now the eigenvalue bound (15.20) contradicts our previous result of Schwarz. □

A theorem of Barbosa and do Carmo type

With our next result from Fröhlich [62] we present some sort of inverse of Schwarz' theorem on μ-stable F-minimal surfaces. Origininally it was proved in Barbosa and do Carmo [6] for minimal surfaces.

Theorem 15.9. *Let the weighted conformally parametrized F-minimal surface $X\colon B\to R^3$ be given. Let $\mathring{B}'\subset\subset\mathring{B}$ denote an open disc, and with $\Omega'\subset S^2$ we mean the spherical domain which is convered by the restriction $N|_{\mathring{B}'}$. For given $\omega_0\geq 0$ we choose an angle $\omega=\omega(\omega_0)\in(0,\pi)$ such that for the spherical cap $S^2_\omega\subset S^2$ it holds*

$$\operatorname{Area}(\Omega')<\operatorname{Area}(S^2_\omega),\quad \lambda_1(S^2_\omega)=\mu(1+\omega_0) \tag{15.21}$$

with a real number $\mu>0$. Then $X|_{\overline{\mathring{B}'}}$ is μ-stabil with this $\mu>0$.

Proof. 1. Assume that $X\colon B\to\mathbb{R}^3$ is not μ-stable. Then on a simply connected and open domain $\mathring{B}^*\subseteq\mathring{B}'$ there is a function $\psi\in C^2(B^*,\mathbb{R})$ satisfying

$$\begin{aligned}\Delta\psi(u,v)+\mu(-K)W\psi(u,v)&=0\quad\text{in } \mathring{B}^*,\\ \psi(u,v)&>0\quad\text{in } \mathring{B}^*,\\ \psi(u,v)&=0\quad\text{on } \partial B^*.\end{aligned}\tag{15.22}$$

Let $\Omega^*\subseteq\Omega'$ be that domain which is covered by N restricted on $\mathring{B}^*\subseteq\mathring{B}'$. We construct a function $\varphi\in\mathscr{V}(\Omega^*,\mathbb{R})$ with the property

$$\int\limits_{\Omega^*}\|\nabla_{S,W}\varphi\|^2_{S,W}\,dS\leq\mu\int\limits_{\Omega^*}\varphi^2\,dS. \tag{15.23}$$

Then our comparison principle yields

$$\int\limits_{\Omega^*}\|\nabla_S\varphi\|^2_S\,dS\leq\mu(1+\omega_0)\int\limits_{\Omega^*}\varphi^2\,dS$$

implying $\lambda_1(\Omega^*)\leq\mu(1+\omega_0)$.

Now let $S^2_{\omega^*} \subset S^2$ be a spherical cap with the same area as $\Omega^* \subset S^2$. Together with (15.21) as well as Proposition 13.4 and 13.3 from chapter 13 we infer

$$\mu(1+\omega_0) = \lambda_1(S^2_\omega) < \lambda_1(S^2_{\omega^*}) \le \lambda_1(\Omega^*) \le \mu(1+\omega_0).$$

Thus we arrive at a contradiction to the μ-stability assumption. It remains to construct $\varphi \in V(\Omega^*, \mathbb{R})$.

2. For some arbitrary point $q \in \overline{\Omega^*}$ we find preimages

$$N^{-1}(q) \cap B^* = \{w_1, \ldots, w_{n_q}\} \subset B^*$$

with multiplicities $\alpha(w_i) \ge 1$, $i = 1, \ldots, n_q$. We set

$$\varphi(q) := \sum_{i=1}^{n_q} \alpha(w_i)\psi(w_i). \tag{15.24}$$

This function is positive on Ω^* and vanishes on the boundary $\partial\Omega^*$ (recall that the spherical mapping is open by Theorem 15.2). Following the argumentation from Barbosa and do Carmo [6], Lemma 3.6, which we skip here, the function is also continuous on the closure of $\Omega^* \subset S^2$. We are left to prove (15.23).

3. Let $\{q_1, \ldots, q_N\} \subset \overline{\Omega^*}$, $N \in \mathbb{N}$, the set of images of branch points in $B^* \subseteq B'$. Choose $\varepsilon > 0$ sufficiently small. In $\overline{\Omega^*} \subseteq \overline{\Omega'}$ we consider closed geodesic discs $\mathfrak{K}_\varepsilon(q_i)$, $i = 1, \ldots, N$, of common radius $\varepsilon > 0$ and centers at q_i for $i = 1, \ldots, N$. Define

$$\mathfrak{T}_\varepsilon := \overline{\Omega^*} \setminus \left\{ \partial\Omega^* \cup \bigcup_{t=1}^{N} \mathfrak{K}_\varepsilon(q_t) \right\}.$$

Following Barbosa and do Carmo [6], Lemma 3.8, there exist simply connected, open and disjoint sets $\mathfrak{R}_{jk} \subset \mathfrak{T}_\varepsilon$ with the following properties:

a) For any point $q \in \mathfrak{R}_{jk}$, the preimage set $N^{-1}(q) \cap \mathring{B}^*$ consists of exactly j elements;

b) it holds

$$\overline{\mathfrak{T}}_\varepsilon = \bigcup_{j,k} \overline{\mathfrak{R}}_{jk}; \tag{15.25}$$

c) there exist simply connected, open and disjoint sets $\mathring{B}^*_{jkl} \subset \mathring{B}^*$ such that

$$N^{-1}(\mathring{R}_{jk}) = \bigcup_{l=1}^{j} \mathring{B}^*_{jkl};$$

thus the spherical mapping restricted on such a $\mathring{B}^*_{jkl} \subset \mathring{B}^*$ is diffeomorphic onto $\mathfrak{R}_{jk} \subset \mathfrak{T}_\varepsilon$.

4. Let $N_{jkl} := N|_{\mathring{B}^*_{jkl}}$. Then the spherical mapping is locally invertible with the inverse mappings
$$N^{-1}_{jkl} : \mathfrak{R}_{jk} \longrightarrow \mathring{B}^*_{jkl}\,, \quad l = 1, \ldots, j. \tag{15.26}$$
Furthermore, the function $\varphi : \overline{\Omega^*} \to \mathbb{R}$ from (15.24) satisfies
$$\varphi|_{\overline{\mathfrak{R}}_{jk}} = \sum_{l=1}^{j} \psi \circ N^{-1}_{jkl}\,. \tag{15.27}$$
It is continuously differentiable in $\overline{\mathfrak{R}}_{jk}$ since $\overline{\mathfrak{R}}_{jk}$ is free of branch points.

5. We are going to prove the inequality
$$\lim_{\varepsilon\to 0} \int_{\mathscr{T}_\varepsilon} \|\nabla_{S,g}\varphi\|^2_{S,g}\,dS \le \lim_{\varepsilon\to 0} \mu \int_{\mathscr{T}_\varepsilon} \varphi^2\,dS. \tag{15.28}$$
The right hand side here is bounded due to the continuity of $\varphi : \overline{\Omega^*} \to \mathbb{R}$, and so is $\varphi \in V(\Omega^*, \mathbb{R})$, and in the limit we obtain
$$\int_{\Omega^*} \|\nabla_{S,W}\,\varphi\|^2_{S,W}\,dS \le \mu \int_{\Omega^*} \varphi^2\,dS.$$
Taking account of (15.25) and (15.27) we compute
$$\begin{aligned}\int_{\mathfrak{T}_\varepsilon} \|\nabla_{S,W}\,\varphi\|^2_{S,W}\,dS &= \sum_{j,k} \int_{\mathfrak{R}_{jk}} \|\nabla_{S,W}\,\varphi\|^2_{S,W}\,dS \\ &= \sum_{j,k} \int_{\mathfrak{R}_{jk}} \left\| \sum_{l=1}^{j} \nabla_{S,W}\,(\psi \circ N^{-1}_{jkl}) \right\|^2_{S,W} dS \\ &= \sum_{j,k} \int_{\mathfrak{R}_{jk}} \sum_{l=1}^{j} \|\nabla_{S,W}\,(\psi \circ N^{-1}_{jkl})\|^2_{S,W}\,dS + \mathfrak{I}_1\end{aligned} \tag{15.29}$$
with the setting
$$\mathfrak{I}_1 := \sum_{j,k} \int_{\mathfrak{R}_{jk}} 2 \sum_{1\le r<s\le j} \langle \nabla_{S,W}\,(\psi \circ N^{-1}_{jkr}), \nabla_{S,W}\,(\psi \circ N^{-1}_{jks}) \rangle_{S,W}\,dS \tag{15.30}$$
where $\langle \cdot, \cdot \rangle$ denotes the inner product on S^2 w.r.t. the weighted metric $(h_{ij})_{i,j=1,2}$. Fom the linear connection of the weighted fundamental forms together with (15.26) we conclude (using weighted conformal parameters)

$$\begin{aligned}
\sum_{l=1}^{j} \int\limits_{\mathfrak{R}_{jk}} \|\nabla_{S,W}(\psi \circ N_{jkl}^{-1})\|_{S,W}^{2}\, dS &= \sum_{l=1}^{j} \iint\limits_{B_{jkl}^{*}} \nabla_{d\sigma_W^2}(\psi,\psi) W_{d\sigma_W^2}\, dudv \\
&= \sum_{l=1}^{j} \iint\limits_{B_{jkl}^{*}} \nabla_{ds_W^2}(\psi,\psi) W\, dudv.
\end{aligned} \tag{15.31}$$

If we set

$$B_{\varepsilon}^{*} := N^{-1}\left(\bigcup_{t=1}^{N} \mathfrak{K}_{\varepsilon}(q_t)\right) \quad with \quad B^{*} \setminus \mathring{B}_{\varepsilon}^{*} = \bigcup_{j,k,l} B_{jkl}^{*}$$

it follows together with (15.29) and (15.31)

$$\begin{aligned}
\int\limits_{\mathfrak{T}_{\varepsilon}} \|\nabla_{S,W}\varphi\|_{S,W}^{2}\, dS &= \sum_{j,k,l} \iint\limits_{B_{jkl}^{*}} \nabla_{ds_W^2}(\psi,\psi) W\, dudv + \mathfrak{I}_1 \\
&= \iint\limits_{B^{*}\setminus B_{\varepsilon}^{*}} \nabla_{ds_W^2}(\psi,\psi) W\, dudv + \mathfrak{I}_1\,.
\end{aligned} \tag{15.32}$$

Consider the right hand side of (15.28):

$$\int\limits_{\mathfrak{T}_{\varepsilon}} \varphi^2\, dS = \sum_{j,k} \int\limits_{\mathscr{R}_{jk}} \left(\sum_{l=1}^{j} \psi \circ N_{jkl}^{-1}\right)^{2} dS = \sum_{j,k}\sum_{l=1}^{j} \int\limits_{\mathscr{R}_{jk}} (\psi \circ N_{jkl}^{-1})^2\, dS + \mathfrak{I}_2$$

due to (15.27) and with the setting

$$\mathfrak{I}_2 := \sum_{j,k} \int\limits_{\mathfrak{R}_{jk}} 2 \sum_{1\le r<s\le j} (\psi \circ N_{jkr}^{-1})(\psi \circ N_{jks}^{-1})\, dS. \tag{15.33}$$

Using weighted conformal parameters this means

$$\begin{aligned}
\int\limits_{\mathfrak{T}_{\varepsilon}} \varphi^2\, dS &= \sum_{j,k,l} \iint\limits_{B_{jkl}^{*}} \psi^2(-K) W\, dudv + \mathfrak{I}_2 \\
&= \iint\limits_{B^{*}\setminus B_{\varepsilon}^{*}} \psi^2(-K) W\, dudv + \mathfrak{I}_2\,.
\end{aligned} \tag{15.34}$$

6. From (15.32), (15.34) and (15.22) we infer

$$\begin{aligned}
&\int\limits_{\mathfrak{T}_\varepsilon} \|\nabla_{S,W}\varphi\|^2_{S,W}\,dS - \mu\int\limits_{\mathfrak{T}_\varepsilon}\varphi^2\,dS \\
&\qquad = \iint\limits_{B^*\setminus B^*_\varepsilon} |\nabla\psi|^2\,dudv - \mu\iint\limits_{B^*\setminus B^*_\varepsilon}\psi^2(-K)W\,dudv + \mathfrak{I}_1 - \mu\mathfrak{I}_2 \\
&\qquad = \iint\limits_{B^*\setminus B^*_\varepsilon} |\nabla\psi|^2\,dudv + \iint\limits_{B^*\setminus B^*_\varepsilon}\psi\triangle\psi\,dudv + \mathfrak{I}_1 - \mu\mathfrak{I}_2 \\
&\qquad = \int\limits_{\partial B^*_\varepsilon}(\psi\nabla\psi)\cdot(N^1,N^2)\,ds + \mathfrak{I}_1 - \mu\mathfrak{I}_2
\end{aligned}$$

with the outer unit normal vector field (N^1,N^2) at the boundary ∂B^*_ε and its line element ds. The function ψ is continuously differentiable in $B^*\subseteq B'$ and vanishes outside of this set such that

$$\lim_{\varepsilon\to 0}\int\limits_{\partial B^*_\varepsilon}(\psi\nabla\psi)\cdot(N^1,N^2)\,ds = 0.$$

Thus it remains to show

$$\lim_{\varepsilon\to 0}(\mathfrak{I}_1-\mu\mathfrak{I}_2)\le 0. \tag{15.35}$$

7. Now let

$$\chi_\ell := \psi\circ N^{-1}_{jkl} : \overline{\mathfrak{R}}_{jk}\longrightarrow\mathbb{R},\quad \ell=1,\ldots,j.$$

Due to (15.27) we have $\varphi|_{\overline{\mathfrak{R}}_{jk}} = \chi_1+\ldots+\chi_j$. Furthermore, there hold

$$\Delta_{S,W}\chi_\ell + \mu\chi_\ell = 0 \quad\text{in}\quad \mathfrak{R}_{jk},\quad \ell=1,\ldots,j,$$

in view of (15.22). We compute

$$\begin{aligned}
\Delta_{S,W}(\chi_r\chi_s) &= \chi_r\Delta_{S,W}\chi_s + \chi_s\Delta_{S,g}\chi_r + 2\,\langle\nabla_{S,W}\chi_r,\nabla_{S,W}\chi_s\rangle_{S,g} \\
&= -2\mu\chi_r\chi_s + 2\,\langle\nabla_{S,W}\chi_r,\nabla_{S,W}\chi_s\rangle_{S,W}\,.
\end{aligned}$$

Together with (15.30) and (15.33) it therefore follows

$$\begin{aligned}
\mathfrak{I}_1-\mu\mathfrak{I}_2 &= \sum_{j,k}\sum_{1\le r<s\le j}\int\limits_{\mathfrak{R}_{jk}}\Big\{2\,\langle\nabla_{S,W}\chi_r,\nabla_{S,W}\chi_s\rangle_{S,W} - 2\mu\chi_r\chi_s\Big\}\,dS \\
&= \sum_{j,k}\sum_{1\le r<s\le j}\int\limits_{\mathfrak{R}_{jk}}\Delta_{S,g}(\chi_r\chi_s)\,dS \\
&= \sum_{j,k}\sum_{1\le r<s\le j}\int\limits_{\partial\mathfrak{R}_{jk}}\langle\nabla_{S,W}(\chi_r\chi_s),\mathfrak{N}\rangle_{S,W}\,d\sigma_S
\end{aligned}$$

with the unit normal vector $\mathfrak{N}$ at the positively oriented boundary curve $\partial\mathfrak{R}_{jk}$ and its line element $d\sigma_S$. Now if $\chi_r\chi_s = 0$ along an arc $\gamma \subset \partial\mathfrak{R}_{jk}$, $\gamma \not\subset \partial\mathfrak{K}_\varepsilon(q_t)$ for $t \in \{1,\ldots,N\}$, then the inner product in the integral is nonositive due to $\chi_r\chi_s > 0$ in $\mathfrak{R}_{jk}$. In case $\chi_r\chi_s \neq 0$ along $\gamma \subset \mathfrak{R}_{jk}$ and $\gamma \not\subset \partial\mathfrak{K}_\varepsilon(q_t)$ for $t \in \{1,\ldots,N\}$, this arc is boundary of two such domains, and we integrate in opposite directions respectively. Now let $\gamma_{jk} \subset \partial\mathfrak{K}_\varepsilon(q_t)$ for $t \in \{1,\ldots,N\}$. We must show

$$\lim_{\varepsilon\to 0}\sum_{j,k}\sum_{1\le r<s\le j}\int\limits_{\gamma_{jk}}\langle\nabla_{S,W}(\chi_r\chi_s),\mathscr{N}\rangle_{S,W}\,d\sigma_S \le 0.$$

Due to the boundness of $\psi = \psi(u,v)$ on $B^* \subseteq \overline{B}'$ it follows

$$\begin{aligned}\Big|\int\limits_{\gamma_{jk}}\langle\nabla_{S,W}(\chi_r\chi_s),\mathscr{N}\rangle_{S,W}\,d\sigma_S\Big| &= \Big|\int\limits_{\gamma_{jk}}\langle\chi_r\nabla_{S,W}\chi_s+\chi_s\nabla_{S,W}\chi_r,\mathscr{N}\rangle_{S,W}\,d\sigma_S\Big| \\ &= \int\limits_{\gamma_{jk}}|\chi_r||\langle\nabla_{S,W}\chi_s,\mathscr{N}\rangle_{S,W}|\,d\sigma_S+\int\limits_{\gamma_{jk}}|\chi_s||\langle\nabla_{S,W}\chi_r,\mathscr{N}\rangle_{S,W}|\,d\sigma_S \\ &= \text{const}\int\limits_{\gamma_{jk}}\Big\{|\langle\nabla_{S,W}\chi_s,\mathscr{N}\rangle_{S,W}|+|\langle\nabla_{S,W}\chi_r,\mathscr{N}\rangle_{S,W}|\Big\}\,d\sigma_S\,.\end{aligned}$$

An arc $\beta := N_{jkr}^{-1}(\gamma_{jk})$ in $\mathring{B} \subset \mathbb{R}^2$ degenerates to a point $\mathring{B}$ for $\varepsilon \to 0$. We get

$$\int\limits_{\gamma_{jk}}\langle\nabla_{S,W}\chi_r,\mathfrak{N}\rangle_{S,W}\,d\sigma_S = \int\limits_{\gamma_{jk}}\langle\nabla_{S,W}(\psi\circ N_{jkr}^{-1}),\mathscr{N}\rangle_{S,W}\,d\sigma_S = \int\limits_{\beta}\nabla\psi\cdot(N^1,N^2)\,dt$$

where we note

$$1 = \langle\mathfrak{N},\mathfrak{N}\rangle_{d\sigma_g^2} = (-K)W|\mathfrak{N}|^2\,,\quad \mathfrak{N} = (N^1,N^2)$$

as well as

$$d\sigma_S = \sqrt{-KW}\,dt.$$

We finally proceed as in point 6 of the proof to show (15.35). The statement follows. □

15.5 A curvature estimate

Application of the previous general curvature estimate

With the considerations so far we are able to realize the assumptions we made for our curvature estimate from section 11.9 in case of F-minimal surfaces with at least special weights $\mathbf{W} = \mathbf{W}(Z)$.

Theorem 15.10. *Let the weighted conformally parametrized and μ-stable F-minimal surface $X\colon B \to \mathbb{R}^3$ be given. Suppose that*

$$\mu > \frac{1+\omega_0}{2}\,.$$

Let furthermore the surface represent a geodesic disc $\mathfrak{B}_r(X_0)$ of radius $r > 0$ with center $X_0 := X(0,0)$. Then there is a constant $\Theta = \Theta(\omega_0, \omega_2, \mu) \in [0, +\infty)$ such that it holds the curvature estimate

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \le \frac{1}{r^2}\,\Theta(\omega_0, \omega_2, \mu)$$

with the principal curvatures κ_1 and κ_2 of the surface.

Proof. We only give a sketch of the proof.

1. From chapter 11, formula (11.33) we first recall

$$|\Delta(r^{-1}X)| \le 2(1+\omega_0)\omega_2 |\nabla(r^{-1}X)||\nabla N|,$$
$$|\Delta N| \le (1+\omega_2)|\nabla N|^2$$

in B. Note that this system is not coupled!

2. From Theorem 15.8 we infer an oscillation estimate for the spherical mapping N such that we can proceed as in the proof of the curvature estimate Theorem 11.13 to obtain a gradient bound of the following form

$$|\nabla N(w_0)| \le k_3(\omega_0, \omega_2, \mu) \quad \text{for all } w_0 \in B_{1-\nu_0}(0,0).$$

Inserting this into the differential inequality for the surface vector linearized the system.

3. The constant $d_0 \in (0, +\infty)$ for the Dirichlet growth estimate

$$\mathscr{D}[X] \le d_0 r^2$$

follows immediately from the previous lemma. Finally, the modulus of continuity for the surface vector X follows from a convex-hull property Theorem 11.12 taking account of $K \le 0$ for the Gaussian curvature K.

A curvature estimate due to Hoy

Following the considerations in Hoy [97] we prove a further curvature estimate for μ-stable W-minimal surfaces assuming a growth condition for the curvature integra which thus replaces the above smallness condition for the stability constant $\mu > 0$.

Theorem 15.11. *Let the μ-stable F-minimal surface $X\colon B \to \mathbb{R}^3$ with $\mu > 0$ be given. Assume that it represents a geodesi disc $\mathfrak{B}_r(X_0)$ of radius $r > 0$ with center $X_0 = X(0,0)$. Then it holds the integral curvature estimate*

$$-\int\limits_0^\sigma\int\limits_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\rho d\varphi \le \frac{1+\omega_0}{\mu\ln\frac{r}{\sigma}}\left\{2\pi - \int\limits_0^r\int\limits_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\rho d\varphi\right\}$$

for all $0 < \sigma \le r$.

Proof. For positive $\sigma \in (0,r]$ we consider the non-negative test function

$$\Phi(\rho) := \begin{cases} \displaystyle\int\limits_\sigma^r \frac{dt}{L(t)} \equiv c_0 \in \mathbb{R} & \text{for all } 0 \le \rho \le \sigma \\ \displaystyle\int\limits_\rho^r \frac{dt}{L(t)} & \text{for } \sigma < \rho \le r \end{cases}$$

with the function

$$L(\rho) = \int\limits_0^{2\pi} \sqrt{P(\rho,\varphi)}\,d\varphi, \quad 0 < \rho \le r$$

satisfying

$$L'(\rho) = 2\pi - \int\limits_0^\rho\int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi,$$

$$L''(\rho) = -\int\limits_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\varphi,$$

see section 14.2. Note that due to $K \le 0$ the derivatives $L'(\rho)$ and $L''(\rho)$ are also non-negative for all $\rho \in [0,r]$. Together with the estimate

$$\int\limits_0^r L''(\rho)\Phi(\rho)^2\,d\rho \le \frac{1+\omega_0}{\mu}\int\limits_0^r \Phi'(\rho)^2 L(\rho)\,d\rho$$

(compare with the derivation of a similar inequality in section 14.2) we compute

$$\begin{aligned} c_0^2\int\limits_0^\sigma L''(\rho)\,d\rho &= \int\limits_0^\sigma L''(\rho)\Phi(\rho)^2\,d\rho \le \int\limits_0^r L''(\rho)\Phi(\rho)^2\,d\rho \\ &\le \frac{1+\omega_0}{\mu}\int\limits_0^r \Phi'(\rho)^2 L(\rho)\,d\rho \le \ldots \end{aligned}$$

$$\begin{aligned} \ldots &= \frac{1+\omega_0}{\mu}\int\limits_0^\sigma \Phi'(\rho)^2 L(\rho)\,d\rho + \frac{1+\omega_0}{\mu}\int\limits_\sigma^r \Phi'(\rho)^2 L(\rho)\,d\rho \\ &= \frac{1+\omega_0}{\mu}\int\limits_\sigma^r \frac{1}{L(\rho)^2} L(\rho)\,d\rho = \frac{1+\omega_0}{\mu}\int\limits_\sigma^r \frac{1}{L(\rho)}\,d\rho \qquad (15.36) \\ &= \frac{(1+\omega_0)c_0}{\mu}\,. \end{aligned}$$

Comparing the graphical shape of the function $L(\rho)$ with a usual theorem of intersecting lines in plane Euclidean geometry we see

$$\frac{\rho}{L(\rho)} \ge \frac{r}{L(r)}$$

for all $\rho \in [\sigma, r]$. Thus we obtain

$$c_0 = \int\limits_\sigma^r \frac{d\rho}{L(\rho)} \ge \int\limits_\sigma^r \frac{r}{L(r)\rho}\,d\rho = \frac{r}{L(r)}\ln\frac{r}{\sigma}\,,$$

and the previous estimate yields

$$\int\limits_0^\sigma L''(\rho)\,d\rho \le \frac{1+\omega_0}{\mu c_0} \le \frac{(1+\omega_0)L(r)}{\mu r \ln\frac{r}{\sigma}}\,.$$

On the other hand, integrating $L'(\rho)$ over $\rho = 0\ldots r$ gives (notice that $L(0)=0$ and $K \le 0$)

$$\frac{L(r)}{r} \le 2\pi - \int\limits_0^r\int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi,$$

such that we infer

$$-\int\limits_0^\sigma L''(\rho)\,d\rho \le \frac{1+\omega_0}{\mu\ln\frac{r}{\sigma}}\left\{2\pi - \int\limits_0^r\int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi\right\}.$$

This proves the statement. □

Remarks

We want to conclude this section with the following remarks.

1. Under the Osserman condition

$$N(u,v)\cdot(0,0,1) \ge \cos\omega \quad \text{for all } (u,v)\in B$$

with an angle $\omega \in (0,\pi)$ as well as the smallness condition

$$-\iint\limits_{B} KW\,dudv \leq e_1 < +\infty$$

Sauvigny in [141], Corollary 1, proves a curvature estimate for F-minimal surfaces $X: B \to \mathbb{R}^3$.

2. Hoy in [97] proves a curvature estimate for immersions with quasiconformal spherical image, so-called *Quasi-minimal surfaces*, under an Osserman condition of the above form. The important integral inequality

$$\int\limits_0^r L''(\rho)\Phi(\rho)^2\,d\rho \leq \text{const}\int\limits_0^r \Phi'(\rho)^2 L(\rho)\,d\rho$$

(and therefore (15.36)) is essentially obtained from the quasiconformality.

15.6 Theorems of Bernstein type

We conclude our considerations with presenting various theorems of Bernstein type for surfaces of minimal surface type with prescibed special weights $\mathbf{W} = \mathbf{W}(Z)$. For the following results we refer the reader to Fröhlich [64].

Complete and μ-stable F-minimal surfaces

First we clarify the terminology μ-stability for complete surfaces.

Definition 15.1. The complete immersion $X: \mathbb{R}^2 \to \mathbb{R}^3$ is called *μ-stable with a real number* $\mu > 0$ if its restriction to an arbitrary compact domain $\Omega \subset \mathbb{R}^2$ is μ-stable with this number.

Now from Theorem 15.10 we immediately infer

Theorem 15.12. *Let $X: \mathbb{R}^2 \to \mathbb{R}^3$ be a complete and μ-stable F-minimal surface with $\mu > \frac{1+\omega_0}{2}$. Then this surface represents a plane.*

Proof. Due to $K \leq 0$ we can introduce geodesic discs $\mathfrak{B}_r(X_0)$ for arbitrary $r > 0$. Using conformal parameters we infer the curvature estimate

$$\kappa_1(\xi,\eta)^2 + \kappa_2(\xi,\eta)^2 \leq \frac{1}{r^2}\,\Theta(\omega_0,\omega_2,\mu).$$

The statement follows from $r \to \infty$. $\square$

Let us focus on some special cases.

Stable F-minimal surfaces

Let $X\colon B\to\mathbb{R}^3$ be a stable and critical point of the parametric functional $\mathscr{F}[X]$. As we have seen in section 15.3, the surface is also μ-stable with $\mu\le 2M_1M_2^{-1}$. The condition $\mu>\frac{1+\omega_0}{2}$ thus leads to

$$2M_1>M_2\,. \tag{15.37}$$

Corollary 15.1. *Let $X\colon B\to\mathbb{R}^3$ be a conformally parametrized and μ-stable F-minimal surface representing a geodesic disc $\mathfrak{B}_r(X_0)$ of radius $r>0$ and with center $X_0\in X(0,0)$. Suppose (15.37) holds true. Then there is a constant $\Theta=\Theta(\omega_0,\omega_2,\mu)\in[0,+\infty)$ such that*

$$\kappa_1(0,0)^2+\kappa_2(0,0)^2\le\frac{1}{r^2}\,\Theta(\omega_0,\omega_2,\mu).$$

If additionally $X\colon\mathbb{R}^2\to\mathbb{R}^3$ is complete then it is affine-linear.

Surfaces with prescribed spherical image

Let $\Omega'\subseteq\Omega\subset S^2$ be covered by the restriction $N|_{\overline{B'}}$. For $\omega_0\ge 0$ let a spherical angle $\omega=\omega(\omega_0)\in(0,+\pi)$ be chosen such that for the spherical cap $S^2_\omega\subset S^2$ there hold

$$\text{Area}\,(\Omega)<\text{Area}\,(S^2_\omega),\quad \lambda_1(S^2_\omega)=\mu(1+\omega_0) \tag{15.38}$$

with a real number $\mu>0$. By the generalized theorem of Barbosa and do Carmo the F-minimal surface $X|_{\overline{B'}}$ is then μ-stable with this number, i.e. we have

$$\iint\limits_{B'}|\nabla\varphi|^2\,dudv\ge\mu\iint\limits_{B'}\varphi^2(-K)W\,dudv\quad\text{for all }\varphi\in C_0^\infty(B',\mathbb{R}).$$

Now this inequality is true for all sets $B'\subset\subset B$, and in each such set there are at most finitely man branch points of the spherical mapping. On the other hand, note that each admissable test function has compact support. Then we conclude

$$\iint\limits_{B}|\nabla\varphi|^2\,dudv\ge\mu\iint\limits_{B}\varphi^2(-K)W\,dudv\quad\text{for all }\varphi\in C_0^\infty(B,\mathbb{R}).$$

Corollary 15.2. *Let $X\colon B\to\mathbb{R}^3$ be a weighted conformally parametrized F-minimal surface representing a geodesic disc $\mathfrak{B}_r(X_0)$. Let furthermore $S^2_\omega\subset S^2$ be a spherical cap satisfying (15.38). Then with a real constant $\Theta=\Theta(\omega_0,\omega_2,\mu)\in[0,+\infty)$, where $\mu:=(1+\omega_0)^{-1}\lambda_1(S^2_\omega)$, it holds the curvature estimate*

$$\kappa_1(0,0)^2+\kappa_2(0,0)^2\le\frac{1}{r^2}\,\Theta(\omega_0,\omega_2,\mu).$$

If the immersion is additionally defined on the whole plane $\mathbb{R}^2$ *and complete then it represents a plane.*

Graphs of minimal surface type

Consider immersions of minimal surface type which can be represented as graphs over the $[x,y]$-plane. Introduce weighted conformal parameters $(u,v) \in B$. We thus get a mapping with the property

$$\text{Area}\,(\Omega) < 2\pi$$

for its spherical image. Recalling $\lambda_1(S^2_{\pi/2}) = 2$ for the first eigenvalue of the spherical Laplacian on the half sphere(see section 13.5)) we set

$$\mu := \frac{2}{1+\omega_0}\,.$$

Moreover, $\mu > \frac{1+\omega_0}{2}$ must be fulfilled.

Corollary 15.3. *Let* $X\colon B \to \mathbb{R}^3$ *be a weighted minimal surface which can be represented as a graph over the* $[x,y]$*-plane and which represents a geodesic disc* $\mathfrak{B}_r(X_0)$*. Suppose that* $0 \le \omega_0 < 1$*. Then there is a real constant* $\Theta = \Theta(\omega_0,\omega_2,\mu) \in [0,+\infty)$*,* $\mu := 2(1+\omega_0)^{-1}$*, such that*

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \le \frac{1}{r^2}\,\Theta(\omega_0,\omega_2,\mu).$$

If the graph is complete then it represents a plane.

Graphs with prescribed growth for the curvatura integra

Let us return to Hoy's curvature estimate.

Corollary 15.4. *Let the complete and* μ*-stable F-minimal surface* $X\colon \mathbb{R}^2 \to \mathbb{R}^3$ *be given. Assume that*

$$\lim_{r\to\infty} \frac{1}{\ln\frac{r}{\sigma}} \int_0^r \int_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\rho\,d\varphi = 0$$

for all $\sigma > 0$*. Then the surface represents a plane.*

Final remarks

1. Our version of this Bernstein-type theorem is somewhat weaker than Hoy's version from [97]. There the reader finds the assumption

$$\lim_{r\to\infty} \frac{1}{\ln^\alpha(1+r)} \int_0^r \int_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\rho\,d\varphi = 0$$

for arbitrary $\alpha \in (0,+\infty)$.

2. Following [97], a growth condition of the form

$$\int_0^r L(\rho)\,d\rho \le d_0 r^2\,, \quad d_0 \in (0,+\infty),$$

with the above function $L = L(\rho)$ already implies the condition

$$\lim_{r\to\infty} \frac{1}{\ln^\alpha(1+r)} \int_0^r \int_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\rho\,d\varphi = 0.$$

Since due to the monotonicity of $L = L(\rho)$ we know

$$L(r)r \le \int_r^{2r} L(\rho)\,d\rho \le \int_0^{2r} L(\rho)\,d\rho \le 4d_0 r^2\,,$$

and therefore $L(r) \le 4d_0 r$. Analogously we infer

$$L'(r)r \le \int_r^{2r} L'(\rho)\,d\rho \le \int_0^{2r} L'(\rho)\,d\rho = L(2r) - L(0) = L(2r) \le 8d_0 r$$

which leads us to $L'(r) \le 8d_0$. Summarizing we get the statement with

$$\frac{\int_0^r L''(\rho)\,d\rho}{\ln^\alpha(1+r)} = \frac{L'(r)-L'(0)}{\ln^\alpha(1+r)} \le \frac{8d_0 - 2\pi}{\ln^\alpha(1+r)}\,.$$

Within our framework such a d_0 is realizable by a a lower bound $\mu > \frac{1+\omega_0}{2}$.

References

1. Abresch, U.
   Constant mean curvature tori in terms of elliptic functions
   Journal für die Reine und Angewandte Mathematik 374, 169–192, 1987
2. Almgren, F.
   Plateau's problem
   W.A. Benjamin, Inc., 1966
3. Alt, W.
   Lineare Funktionalanalysis
   Springer, 2006
4. Bär, C.
   Elementare Differentialgeometrie
   Walter de Gruyter, 2001
5. Bandle, C.
   Isoperimetric inequalities and applications
   Pitman Advanced Publishing Program, 1980
6. Barbosa, J.L.; do Carmo, M.
   On the size of stable minimal surfaces in $\mathbb{R}^3$
   The American Journal of Mathematics 98, 515–528, 1974
7. Barbosa, J.L.; do Carmo, M.
   A proof of a general isoperimetric inequality for surfaces
   Mathematische Zeitschrift 162, 245–261, 1978
8. Barbosa, J.L.; do Carmo, M.
   Stability of minimal surfaces and eigenvalues of the Laplacian
   Mathematische Zeitschrift 173, 13–28, 1980.
9. Beckenbach, E.F.; Radó, T.
   Subharmonic functions and surfaces of negative curvature
   Transactions of the American Mathematical Society 35, 662–674, 1933
10. Begehr, H.G.W.; Wen, G.C.
    Nonlinear boundary value problems and their applications
    Addison Wesley Longman Ltd., 1990
11. Bergner, M.
    Konstruktion von Flächen vorgeschriebener mittlerer Krümmung mit der Kontinuitätsmethode
    Dissertation, Darmstadt, 2006

12. Bergner, M.; Dittrich, J.
On surfaces of prescribed weighted mean curvature
Calculus of Variations and Partial Differential Equations 33, 169–185, 2008
13. Bergner, M.; Fröhlich, S.
On two-dimensional immersions of prescribed mean curvature in $\mathbb{R}^n$
Zeitschrift für Analysis und ihre Anwendungen 27, No. 1, 31–52, 2008
14. Bers, L.
Univalent solutions of Linear elliptic systems
Communications on Pure and Applied Mathematics VI, 513–526, 1953
15. Blaschke, W.; Leichtweiss, K.
Elementare Differentialgeometrie
Springer, 1973
16. Böhme, R.; Hildebrandt, S.; Tausch, E.
The two-dimensional analogue of the catenary
Pacific Journal of Mathematics 88, No. 2, 1980
17. Bourgain, J.; Brezis, H; Mironescu, P.
Lifting in Sobolev spaces
Journal d'Analyse Mathematique 80, 37–86, 2000
18. Brauner, H.
Differentialgeometrie
Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, 1981
19. Brezis, H.; Coron, J.M.
Multiple solutions of H-systems and Rellich's conjecture
Communications on Pure and Applied Mathematics 37, 149–187, 1984
20. Burago, Y.D.; Zalgaller, V.A.
Geometric inequalities
Springer, 1988.
21. do Carmo, M.P.
Differential geometry of curves and surfaces
Prentice-Hall, Inc., 1976
22. do Carmo, M.P.
Riemannian geometry
Birkhäuser, 1992
23. Cartan, H.
Differentialformen.
BI Wissenschaftsverlag, 1974.
24. Cesari, L.
An existence theorem of the calculus of variations for integrals on parametric surfaces
The American Journal of Mathematics 74, 265–295, 1952
25. Chavel, I.
Isoperimetric inequalities
Cambridge University Press, 2001
26. Chen, B.-Y.
Surfaces with parallel mean curvature vector
Bulletin of the American Mathematical Society 78, No. 5, 709–710, 1972
27. Chen, B.-Y.
Geometry of submanifolds
Marcel Dekker, Inc., 1973
28. Chern, S.-S.
An elementary proof of the existence of isothermal parameters on a surface
Proceedings of the American Mathematical Society 6, No. 5, 771–782, 1955

29. Chern, S.-S.; Hartman, P.; Wintner, A.
On isothermic coordinates
Commentarii Mathematici Helvetici 28, 301–309, 1954

30. Clarenz, U.
Enclosure theorems for extremals of elliptic parametric functionals
Calculus of Variations and Partial Differential Equations 15, 313–324, 2002

31. Clarenz, U.
A stability criterion for extremals of elliptic parametric functionals
Pacific Journal of Mathematics 208, No. 2, 231–242, 2003

32. Clarenz, U.; von der Mosel, H.
On surfaces of prescribed F-mean curvature
Pacific Journal of Mathematics 123, 15–36, 2004

33. Cluttberbuck, J.
Interior gradient estimates for anisotropic mean curvature flow
Pacific Journal of Mathematics 229, No. 1, 119–136, 2007

34. Coifman, R.; Lions, P.-L.; Meyer, Y.; Semmes, S.
Compensated compactness and Hardy spaces
Journal de Mathematique Pures et Appliquees 9, No. 3, 247–286, 1993

35. Colding, T.H.; Minicozzi, W.P.
Minimal surfaces
Courant Institute of Mathematical Sciences, 1999

36. Colding, T.H.; Minicozzi, W.P.
Estimates for parametric elliptic integrands
International Mathematics Research Notices 4, 291–297, 2002

37. da Costa, R.C.T.
Constraints in quantum mechanics
Physical Review A 25, No. 6, 2893–2900, 1982

38. Courant, R.
Dirichlet's principle, conformal mappings, and minimal surfaces
Interscience publishing, Inc., New York, 1950

39. Courant, R.; Hilbert, D.
Methoden der mathematischen Physik
Springer, 1993

40. Courant, R.; Hilbert, D.
Methods of mathematical physics 2
John Wiley & Sons, Inc., 1962

41. Danskin, J.M.
On the existence of minimizing surfaces in parametric double integral problems in the calculus of variations
Rivista di Matematica della Universita di Parma 3, 43–63, 1952.

42. Dierkes, U.
Maximum principles and nonexistence results for minimal surfaces
manuscripta mathematica 69, 203–218, 1990

43. Dierkes, U.
Maximum principles for submanifolds of arbitrary codimension and bounded mean curvature
Calculus of variations and partial differential equations 22, 173–184, 2005

44. Dierkes, U.; Hildebrandt, S.; Küster, A.; Wohlrab, O.
Minimal surfaces
Springer, 1992

45. Dobriner, H.
Die Flächen constanter Krümmung mit einem System sphärischer Krümmungslinien dargestellt mithilfe von Thetafunctionen
Acta Mathematics 9, 73–104, 1886

46. Dobrowolski, M.
Angewandte Funktionalanalysis
Springer, 2005

47. Douglas, J.
Solution of the problem of Plateau
Transactions of the American Mathematical Society 33, No. 1, 263–321, 1931

48. Douglas, J.
The higher topological form of the problem of Plateau
Annali della Scuola Normale Superiore di Pisa (2) 8, 195–218, 1939

49. Douglas, J.
The most general form of the problem of Plateau
The American Journal of Mathematics 61, 59–608, 1939

50. Ecker, K.
Regularity theory for mean curvature flow
Birkäuser, 2004

51. Enmoto, K.
Umbilical points on surfaces in $\mathbb{R}^N$
Nagoya Mathematics Journal 100, 135–143, 1985

52. Eschenburg, J.-H.; Jost, J.
Differentialgeometrie und Minimalflächen
Springer, 2007

53. Evans, L.C.
Partial Differential equations
Oxford University Press, 1998

54. Federer, H.; Fleming, W.H.
Normal and integral currents
Annals of Mathematics 72, 458–520, 1960

55. Fefferman, C.; Stein, E.-M.
H^p *spaces of several variables*
Acta Mathematica 129, 137–193, 1972

56. Fenchel, W.
Über Krümmung und Windung geschlossener Raumkurven
Mathematische Annalen 101, 238–252, 1929

57. Ferapontov, E.V.
Surfaces with flat normal bundle: An explicit construction
Differential geometry and its Applications 14, No. 1, 15–37, 2001

58. Finn, R.
On equations of minimal surface type
Annals of Mathematics 60, No.3, 1954

59. Finsler, P.
Über Kurven und Flächen in allgemeinen Räumen
Birkhäuser, 1951

60. Fischer-Colbrie, D.; Schoen, R.
The structure of complete stable minimal surfaces in 3*-manifolds of non-negative scalar curvature*
Communications on Pure and Applied Mathematics 33, 199–211, 1980

61. Fomin, S.V.; Gelfand, I.M.
Calculus of variations
Prentice-Hall, Inc., 1963

62. Fröhlich, S.
Curvature estimates for μ-stable G-minimal surfaces and theorems of Bernstein type
Analysis 22, 109–130, 2002

63. Fröhlich, S.
On 2-surfaces in $\mathbb{R}^4$ and R^n
Proceedings of the 5th Conference of Balkan Society of Geometers, Mangalia 2005

64. Fröhlich, S.
Über zweidimensionale elliptische Systeme der geometrischen Analysis
Habilitation thesis, Technical University Darmstadt, 2008

65. Fröhlich, S.; Müller, F.
On critical normal sections for two-dimensional immersions in $\mathbb{R}^4$ and a Riemann-Hilbert problem
Differential Geometry and its Applications 26, No. 5, 508–513, 2008

66. Fröhlich, S.; Müller, F.
On critical normal sections for two-dimensional immersions in $\mathbb{R}^{n+2}$
Calculus of Variations and Partial Differential Equations 35, No. 4, 497–515, 2009

67. Fröhlich, S.; Winklmann, S.
Curvature estimates for graphs with prescribed mean curvature and flat normal bundle
manuscripta mathematica 122, 149–162, 2007

68. Funk, P.
Variationsrechnung und ihre Anwendungen in Physik und Technik
Springer, 1962

69. Garnier, R.
Le problém de Plateau
Annales Scientifiques de l'Ecole Normale Superieure (3) 45, 53–144, 1928

70. Gauss, C.F.
General investigations of curved surfaces
Dover Publications, Inc., 2005

71. Gilbarg, D.; Trudinger, N.S.
Elliptic Partial Differential Equations of Second Order
Springer, 1983

72. Giusti, E.
Minimal surfaces and functions of bounded variation
Birkhäuser, 1984

73. Glaeser, G.; Polthier, K.
Bilder der Mathematik
Spektrum Akademischer Verlag, 2009

74. Grassmann, H.
Die neuere Algebra und die Ausdehnungslehre
Mathematische Annalen 7, Nr. 4, 1874

75. Gulliver, R.
Existence of surfaces with prescribed mean curvature vector
Mathematische Zeitschrift 131, 117–140, 1973

76. Gulliver, R.
Minimal surfaces of finite index in manifolds of positive scalar curvature
In: Calculus of Variations and partial differential equations, S. Hildebrandt, D. Kinderlehrer, M. Miranda, Ed., Springer, 1988

77. Hartman, P.; Wintner, A.
On the local behavior of solutions of non-parabolic partial differential equations
The American Journal of Mathematics 75, 449–476, 953

78. Heil, E.
Differentialformen und Anwendungen auf Vektoranalysis, Differentialgleichungen, Geometrie
BI Wissenschaftsverlag, 1974

79. Heinz, E.
Über Flächen mit eineindeutiger Projektion auf eine Ebene, deren Krümmungen durch Ungleichungen eingeschränkt sind
Mathematische Annalen 129, 451–454, 1955

80. Heinz, E.
On certain nonlinear elliptic differential equations and univalent mappings
J. d'Analyse Mathematique 5, 197–272, 1957

81. Heinz, E.
Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern
Mathematische Zeitschrift 113, 99–105, 1970

82. Hélein, F.
Harmonic maps, conservation laws and moving frames
Cambridge University Press, 2002

83. Hellwig, G.
Partial differential equations
B.G. Teubner, 1977

84. Hildebrandt, S.
Einige Bemerkungen über Flächen beschränkter mittlerer Krümmung
Mathematische Zeitschrift 159, 169–178, 1970

85. Hildebrandt, S.
Über einen neuen Existenzsatz für Flächen vorgeschriebener mittlerer Krümmung
Mathematische Zeitschrift 119, 267–272, 1971

86. Hildebrandt, S.; von der Mosel, H.
On two-dimensional parametric variational problems
Calculus of variations and partial differential equations 9, 1999, 249–267

87. Hildebrandt, S.; von der Mosel, H.
Plateau's problem for parametric double integrals: Part I. Existence and regularity in the interior
Communications on Pure and Applied Mathematics 56, 926–955, 2003

88. Hildebrandt, S.; von der Mosel, H.
Plateaus's problem for parametric double integrals: Part II. Regularity at the boundary
Journal für die reine und angewandte Mathematik 565, 207–233, 2003

89. Hildebrandt, S.; von der Mosel, H.
Dominance functions for parametric Lagrangians
In: Geometric analysis and nonlinear partial differential equations, Springer, 297–326, 2003

90. Hildebrandt, S.; von der Mosel, H.
On Lichtenstein's theorem about globally conformal mappings
Calculus of variations and partial differential equations 23, 415–424, 2005

91. Hildebrandt, S.; von der Mosel, H.
Conformal representation of surfaces, and Plateau's problem for Cartan functionals
Rivista di Matematica della Universita di Parma 7, No. 4, 1–43, 2005

92. Hoffman, D.A.
Surfaces in constant curvature manifolds with parallel mean curvature vector field
Bulletin of the American Mathematical Society 78, No. 2, 247–250, 1972

93. Hoffman, D.A.; Osserman, R.
Geometry of the generalized Gauss map
Memoirs of the American Mathematical Society 28, No. 236, 1980.

94. Hopf, H.
Über Flächen mit einer Relation zwischen den Hauptkrümmungen
Mathematische Nachrichten 4, 232–249, 1950/51

95. Hoy, E.
Über Flächen mit quasikonformer Gaußscher Abbildung
Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg, Mathematisch-Naturwissenschaftliche Reihe 38, No. 4, 123–129, 1989

96. Hoy, E.
Zum Bernsteinschen Satz für Quasiminimalflächen
Zeitschrift für Analysis und ihre Anwendungen 10, No. 4, 569–572, 1991

97. Hoy, E.
On Bernstein's theorem for quasiminial surfaces II
Zeitschrift für Analysis und ihre Anwendungen 16, No. 4, 1027–1032, 1997

98. Hurwitz, A.
Quelques applications geometriques des series de Fourier
Annales de l'École Normale 19, No. 3, 357–408, 1902

99. Jenkins, H.B.
On two-dimensional variational problems in parametric form
Archive for Rational Mechanics and Analysis 8, 181–206, 1961

100. Jorge, L.P.; Tomi, F.
The barrier principle for minimal submanifolds of arbitrary codimension
Annals of Global Analysis and Geometry 24, 261–267, 2003

101. Jost, J.
Two-dimensional variations problems
Wiley Interscience, 1991

102. Jost, J.
Riemannian geometry and geometric analysis
Springer, 1995

103. Jost, J.
Partielle Differentialgleichungen
Springer, 1998

104. Jost, J.; Xin, Y.L.
Bernstein type theorems for higher codimension
Calculus of variations and Partial differential equations 9, No. 4, 277–296, 1999

105. Kawai, S.
A theorem of Bernstein type for minimal surfaces in $\mathbb{R}^4$
Tohoku Mathematical Journal 36, 377–384, 1984

106. Klingenberg, W.P.A.
Klassische Differentialgeometrie
Edition am Gutenbergplatz, 2004

107. Koiso, M.; Palmer, B.
Geometry and stability of surfaces with constant anisotropic mean curvature
Indiana Univiversity Mathematics Journal 54, 1817–1852, 2005

108. Koiso, M.; Palmer, B.
Stability of anisotropic capillary surfaces between two parallel planes
Calculus of variations and partial differential equations 25, 275–298, 2006

109. Kühnel, W.
Differentialgeometrie
Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, 1999

110. Kurzke, M.; von der Mosel, H.
The Douglas problem for parametric double integrals
manuscripta mathematica 110, 93–114, 2003

111. Laugwitz, D.
Differentialgeometrie
B.G. Teubner, 1977

112. Lawson, H.B.
Lectures on minimal submanifolds
Publish or Perish, Inc., 1980

113. Liebmann, H.
Eine neue Eigenschaft der Kugel.
Nachrichten der Königlichen Gesellschaft Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 44-55, 1899

114. Liebmann, H.
Ueber die Verbiegung der geschlossenen Flächen positiver Krümmung
Mathematische Annalen 53, 91–112, 1900

115. Lin, F.H.
Estimates for surfaces which are stationary for an elliptic parametric integral
Journal of Partial Differential Equations 3, 78–92, 1990

116. Liu, X.; Terng, C.-L.
The mean curvature flow for isoparametric submanifolds
Duke Mathematical Journal 147, 157–179, 2009

117. Lonseth, A.T.
The problem of Plateau in hyperbolic spaces
The American Journal of Mathematics 64, No. 1, 229–259, 1942

118. Maz'ya, V. G.
Classes of domains and imbedding theorems for function spaces
Soviet Mathematics Doklady 1, 882–885, 1960

119. Michael, J.H.; Simon, L.M.
Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$
Communications on Pure and Applied Mathematics XXVI, 361–379, 1973

120. Morgan, F.
Geometric measure theory: A beginner's guide
Academic Press Inc., 2000

121. Morrey, C.B.
The problem of Plateau on a Riemannian manifold
Annals of Mathematics 49, 807–851, 1948

122. Morrey, C.B.
The parametric variational problem for double integrals
Communications on Pure and Applied Mathematics 14, 569–575, 1961

123. Morrey, C.B.
Multiple integrals in the calculus of variations
Springer, 1966

124. Müller, F.; Schikorra, A.
Boundary regularity via Uhlenbeck-Riviere decomposition
Analysis 29, 199–220, 2009

125. Neutsch, W.
Koordinaten. Theorie und Anwendungen
Spektrum Akademischer Verlag, 1995

126. Nitsche, J.C.C.
Vorlesungen über Minimalflächen
Springer, 1975

127. Osserman, R.
Global properties of minimal surfaces in E^3 and E^n
Annals of Mathematics 80, No. 2, 340–364, 1964

128. Osserman, R.
A proof of the regularity everywhere of the classical solution to Plateau's problem
Annals of Mathematics 91, 550–569, 1970.

129. Osserman, R.
The isoperimetric inequality
Bulleting of the American Mathematical Society 84, No. 6, 1978

130. Osserman, R.
A survey of minimal surfaces
Dover Publications, Inc., 1986

131. Osserman, R.; Ru, M.
An estimate for the Gauss curvature of minimal surfaces in R m whose Gauss map omits a set of hyperplanes
Journal of Differential Geometry 46, 578–593, 1997

132. Pozzi, P.
Anisotropic mean curvature flow in higher codimension
PAMM 8, No. 1, 10849–10850, 2009.

133. Radó, T.
On Plateau's problem
Annals of Mathematics 31, No. 2, 457–469, 1930

134. Räwer, K.
Stabile Extremalen parametrischer Doppelintegrale im $\mathbb{R}^3$
Dissertation, Bonn, 1993

135. Raschewski, P.K.
Riemannsche Geometrie und Tensoranalysis
Verlag Harri Deutsch, 1995

136. Rivière, T.
Conservation laws for conformally invariant variational problems
Inventiones Mathematicae 168, 1–22, 2007.

137. Ruchert, H.
Ein Eindeutigkeitssatz für Flächen konstanter mittlerer Krümmung
Archiv der Mathematik 33, 91–104, 1979

138. Sakai, T.
Riemannian geometry
Translation of mathematical monographs 149, AMS, 1996

139. Sato, S.
Barta's inequality and the first eigenvalue of a cap domain of a 2*-sphere*
Mathematische Zeitschrift 181, No. 3, 313–318, 1982

140. Sauvigny, F.
A-priori-Abschätzungen der Hauptkrümmungen für Immersionen vom Mittleren-Krümmungs-Type mittels Uniformisierung und Sätze vom Bernstein-Typ
Habilitationsschrift, Göttingen, 1989

141. Sauvigny, F.
Curvature estimates for immersions of minimal surface type via uniformization and theorems of Bernstein type
manuscripta mathematica 67, 567–582, 1990

142. Sauvigny, F.
Introduction of isothermal parameters into a Riemannian metric by the continuity method
Analysis 19, No. 3, 235–243, 1999

143. Sauvigny, F.
Partielle Differentialgleichungen der Geometrie und der Physik
Springer, 2004/2005

144. Schauder, J.
Über das Dirichletsche Problem im Großen für nicht-lineare elliptische Differentialgleichungen
Mathematische Zeitschrift 37, 623–634, 1933

145. Schoen, R.; Simon, L.; Yau, S.T.
Curvature estimates for minimal hypersurfaces
Acta Mathematica 134, 275–288, 1975

146. Schulz, F.
Regularity theory for quasilinear elliptic systems and Monge-Amperé equations in two dimensions
Springer, 1990

147. Simon, L.
A Hölder estimate for quasiconformal maps between surfaces in Euclidean space
Acta Mathematica 139, 19–51, 1977

148. Smoczyk, K.; Wang, G.; Xin, Y.L.
Mean curvature flow with flat normal bundle
arXiv:math/0411010v1, 2004

149. Smoczyk, K.; Wang, G.; Xin, Y.L.
Bernstein type theorems with flat normal bundle
Calculus of variations and partial differential equations 26, No. 1, 2006

150. Spruck, J.
Remarks on the stability of minimal submanifolds in $\mathbb{R}^n$
Mathematische Zeitschrift 144, 169–174, 1975

151. Stein, E.M.
Harmonic Analysis: Real-variable methods, orthogonality, and oscillatory integrals
Princeton University Press, 1993

152. Strubecker, K.
Differentialgeometrie
Walter de Gruyter & Co., 1964/1969

153. Taylor, J.E.
Geometric models of crystal growth
Acta Metallurgica et Materialia 40, No. 7, 1443–1474, 1992

154. Taylor, J.E.
Mean curvature and weighted mean curvature
Acta Metallurgica et Materialia 40, No. 7, 1475–1485, 1992

155. Taylor, M.E.
Partial differential equations III: Nonlinear equations
Springer, 2004

156. Terng, C.-L.
Isoparametric submanifolds and their Coxeter group
Journal Differential of Geometry 21, 79–107, 1985

157. Tomi, F.
Ein einfacher Beweis eines Regularitätssatzes für schwache Lösungen gewisser elliptischer Systeme
Mathematische Zeitschrift 112, 214–218, 1969

158. Topping, P.
Mean curvature flow and geometric inequalities
Journal für die reine und angewandte Mathematik 503, 47–61, 1988

159. Topping, P.
The optimal constant in Wente's L^∞-estimate
Commentarii Mathematici Helvetici 72, 316–328, 1997

160. Vekua, I.N.
Systeme von Differentialgleichungen vom elliptischen Typus und Randwertaufgaben
Deutscher Verlag der Wissenschaften, 1956.

161. Vekua, I. N.
Verallgemeinerte analytische Funktionen
Akademie-Verlag, 1963

162. Wang, M.-T.
Stability and curvature estimates for minimal graphs with flat normal bundles
arXiv:math/0411169v2, 2004

163. *Weisstein, E.*
Wolfram MathWorld
Created, developed, and nurtured by Eric Weisstein with contributions from the world's mathematical community

164. Wendland, W.L.
Elliptic systems in the plane
Pitman Publishing Ltd., 1979

165. Wente, H. C.
The differential equation $\Delta x = 2Hx_u \wedge x_v$ with vanishing boundary values
Proceedings of the American Mathematical Society 50, 131-137, 1975

166. Wente, H.C.
Large solutions to the volume constrained Plateau problem
Archive for Rational Mechanics and Analysis 75, 59–77, 1980

167. Wente, H.C.
Counterexample to a conjecture of H. Hopf
Pacific Journal of Mathematics 121, No. 1, 193–243, 1986

168. Weyl, H.
Zur Infinitesimalgeometrie: p-dimensionale Flächen im n-dimensionalen Raum
Mathematische Zeitschrift 12, 154–160, 1922

169. Weyl, H.
Space-Time-Matter
Dover Publications, Inc., 1952

170. White, B.
Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals
Inventiones Mathematicae 88, 243–256, 1987

171. White, B.
Existence of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on 3*-manifolds*
Journal of Differential Geometry 33, 413–443, 1991

172. Winklmann, S.
Integral curvature estimates for F-stable hypersurfaces
Calculus of variations and partial differential equations 23, 391–414, 2005

173. Winklmann, S.
Pointwise curvature estimates for F-stable hypersurfaces
Annales de l'Institut Henry Poincaré (C) Analyse Non Linéaire 22, 543–555, 2005

174. Wulff, G.
Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflächen
Zeitschrift für Krystallographie und Mineralogie 34, 449–530, 1901

175. Xin, Y.L.; Yang, L.
Curvature estimates for minimal submanifolds of higher codimension
arXiv:0709.3686, 2007

176. Yau, S.-T.:
Submanifolds with constant mean curvature I.
The American journal of mathematics 96, No. 2, 346–366, 1974.

List of Names

Index